6.003 TA Office Hour Notes
Spring 2010
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Many of you had questions about the solution to the difference equation of the form

$$y[n] = \alpha y[n-1] + \beta \tag{1}$$

You can use recursion to solve for y[n]

$$\begin{aligned}
y[n] &= \alpha y[n-1] + \beta && (2)\\
&= \alpha(\alpha y[n-2] + \beta) + \beta = \alpha^2 y[n-2] + \alpha\beta + \beta && (3)\\
&= \alpha(\alpha^2 y[n-3] + \alpha\beta + \beta) + \beta = \alpha^3 y[n-3] + \alpha^2\beta + \alpha\beta + \beta && (4)\\
&= \alpha^n y[0] + \alpha^{n-1}\beta + \alpha^{n-2}\beta + ... + \alpha\beta + \beta && (5)\\
&= \alpha^n y[0] + \beta \sum_{k=0}^{n-1} \alpha^k && (6)\\
&= \alpha^n y[0] + \beta \frac{1-\alpha^n}{1-\alpha} && (7)\\
&= \alpha^n \left(y[0] - \frac{\beta}{1-\alpha} \right) + \frac{\beta}{1-\alpha} && (8)\\
& && (9)
\end{aligned}$$

Notice this is of the form

$$y[n] = A\alpha^n + B \tag{10}$$

In HW1 Problem 7 some of you did something similar to the calculation above using a table and noticing a pattern (this was a bit simpler since $\beta = 1$ and $y[0] = 1$ in the HW), or you might have assumed the answer would be of this form, using boundary conditions to solve for A and B (with $|\alpha| < 1$). These produce equivalent results.

$$y_\infty = \alpha y_\infty + \beta \tag{11}$$

$$y_\infty = \frac{\beta}{1-\alpha} \tag{12}$$

$$y[0] = A + B \tag{13}$$

$$y_\infty = \frac{\beta}{1-\alpha} = B \tag{14}$$

$$A = y[0] - \frac{\beta}{1-\alpha} \tag{15}$$

So again,

$$y[n] = \alpha^n \left(y[0] - \frac{\beta}{1-\alpha} \right) + \frac{\beta}{1-\alpha} \tag{16}$$

Yet another way to think about this solution parallels the homogeneous/inhomogeneous solution of a differential equation, with $y[n] = y[0]\alpha^n$ the homogeneous solution, i.e. the solution of the difference equation

$$y[n] = \alpha y[n-1] \tag{17}$$