6.003 (Spring 2010)

Final Examination *May 20, 2010*

Name:

Kerberos Username:

Please circle your section number:

Section	*Instructor*	*Time*
1	Peter Hagelstein	10 am
2	Peter Hagelstein	11 am
3	Rahul Sarpeshkar	1 pm
4	Rahul Sarpeshkar	2 pm

Grades will be determined by the correctness of your answers (explanations are not required).

Partial credit will be given for ANSWERS that demonstrate some but not all of the important conceptual issues.

You have **three hours.**

Please put your initials on all subsequent sheets.

Enter your answers in the boxes.

This quiz is closed book, but you may use four 8.5×11 sheets of paper (eight sides total).

No calculators, computers, cell phones, music players, or other aids.

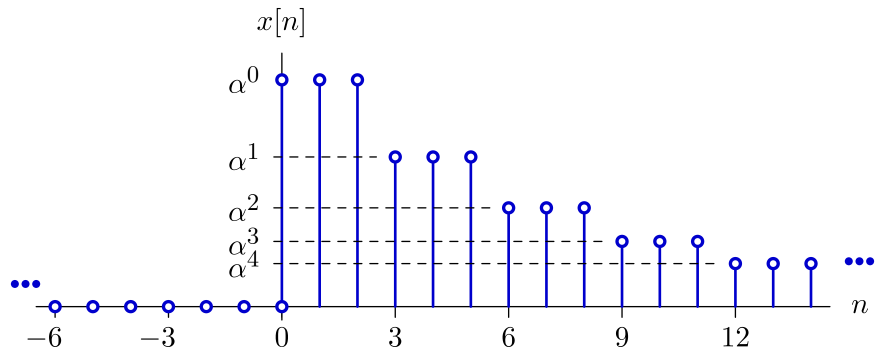
1	/10
2	/12
3	/18
4	/15
5	/15
6	/15
7	/15
Total	/100

1. Z Transform *[10 points]*

Find the Z transform of $x[n]$ defined as

$$x[n] = \alpha^{\lfloor n/3 \rfloor}\, u[n]$$

where α represents a positive real number, $u[n]$ represents the unit-step function, and $\lfloor n/3 \rfloor$ represents the largest integer that is $\leq n/3$. The signal $x[n]$ is plotted below.



Enter a closed-form expression for $X(z)$ in the box below.

$$X(z) = \boxed{\frac{1 + z^{-1} + z^{-2}}{1 - \alpha z^{-3}}}$$

Enter the region of convergence of $X(z)$ in the box below.

$$\text{ROC} = \boxed{|z| > |\sqrt[3]{\alpha}|}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{\substack{n=0 \\ n \text{ divisible by } 3}}^{\infty} \left(\alpha^{n/3} z^{-n} + \alpha^{n/3} z^{-n-1} + \alpha^{n/3} z^{-n-2}\right)$$

$$= \sum_{m=0}^{\infty} \alpha^m z^{-3m} \left(1 + z^{-1} + z^{-2}\right) = \frac{1 + z^{-1} + z^{-2}}{1 - \alpha z^{-3}}$$

where this sum converges iff $|\alpha z^{-3}| < 1$. Thus the region of convergence is $|z| > |\sqrt[3]{\alpha}|$.

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2. CT System Design *[12 points]*

We wish to design a linear, time-invariant, continuous-time system that is causal and stable. For asymptotically low frequencies, the magnitude of the system's frequency response should be 4ω. For asymptotically high frequencies, the magnitude of the system's frequency response should be $100/\omega$.

Is it possible to design such a system so that the magnitude of its frequency response is 50 at $\omega = 5$?

Yes or **No**: Yes

If **Yes**, determine the poles of the resulting system.

poles: $-1 \pm j\sqrt{24}$

If **No**, briefly explain why not.

The asymptotic responses can be achieved with a zero at $\omega = 0$ and two poles in the left half-plane, so that the system function has the form

$$H(s) = \frac{100s}{s^2 + \alpha s + 25}\,.$$

Also, the magnitude at $s = j5$ is 50:

$$H(j5) = \frac{100 \times j5}{-25 + \alpha j5 + 25} = \frac{100}{\alpha} = 50\,.$$

Therefore $\alpha = 2$. Thus the poles are the roots of

$$s^2 + 2s + 25$$

which are $s = -1 \pm \sqrt{1 - 25} = -1 \pm j\sqrt{24}$.

Worksheet (intentionally blank)

3. Frequency Responses *[18 points]*

Match each system function in the left column below with the corresponding frequency response magnitude in the right column.

Enter label of corresponding frequency response magnitude (**A**-**F** or **none**) in box.

$$H_1(z) = \frac{z^3}{z^3 - 0.5} \rightarrow \boxed{\text{F}}$$

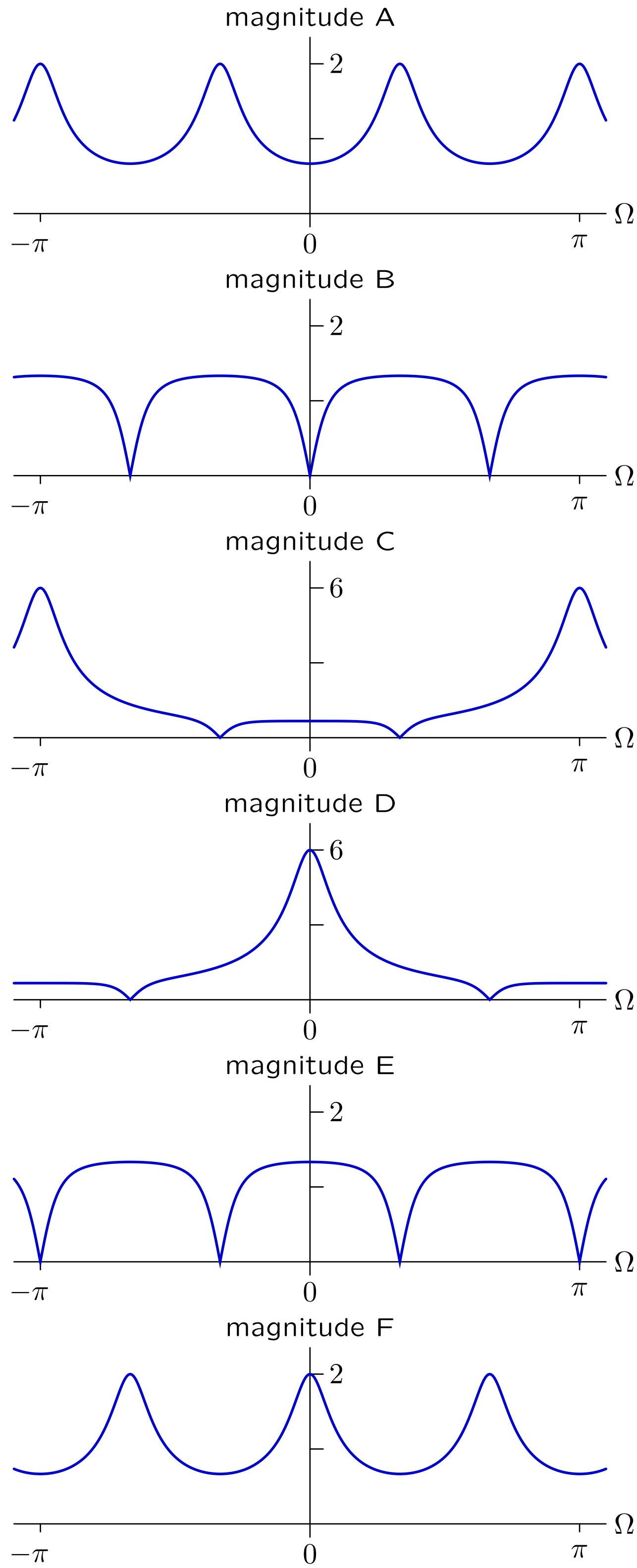
$$H_2(z) = \frac{z^3 - 1}{z^3 - 0.5} \rightarrow \boxed{\text{B}}$$

$$H_3(z) = \frac{z^3 + z^2 + z}{z^3 - 0.5} \rightarrow \boxed{\text{D}}$$

$$H_4(z) = \frac{z^3}{z^3 + 0.5} \rightarrow \boxed{\text{A}}$$

$$H_5(z) = \frac{z^3 + 1}{z^3 + 0.5} \rightarrow \boxed{\text{E}}$$

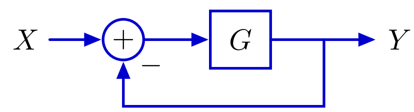
$$H_6(z) = \frac{z^3 - z^2 + z}{z^3 + 0.5} \rightarrow \boxed{\text{C}}$$



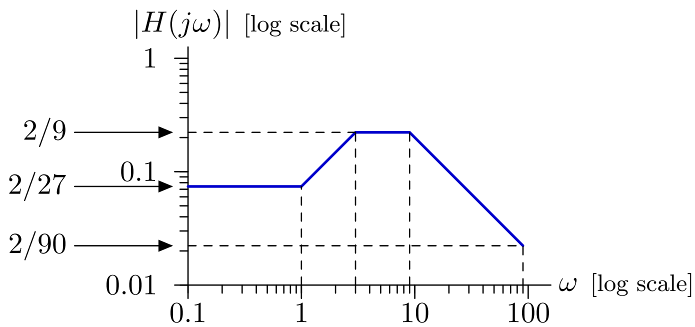
Worksheet (intentionally blank)

4. Feedback Design *[15 points]*

Consider a feedback system of the following form



where G represents a causal, linear, time-invariant, continuous-time system. The magnitude of the frequency response of $H = \frac{Y}{X}$ is specified by the straight-line approximation shown below.



The magnitude for asymptotically low frequencies is $\frac{2}{27} \approx 0.074$.

Determine $G(s)$. [You only need to find one solution, even if others exist.]

$$G(s) = \boxed{\frac{2(s+1)}{(s+5)^2} \text{ or } \frac{2(s-1)}{s^2+10s+29} \text{ or } \frac{-2(s+1)}{s^2+14s+29} \text{ or } \frac{-2(s-1)}{s^2+14s+25}}$$

The system function for H can be determined from the magntitude plot to be

$$H(s) = \frac{2(s+1)}{(s+3)(s+9)}\,.$$

The relation between G and H is given by Black's equation

$$H = \frac{G}{1+G}\,.$$

This relation can be solved for G to yield

$$G = \frac{H}{1-H} = \frac{\frac{2(s+1)}{(s+3)(s+9)}}{1-\frac{2(s+1)}{(s+3)(s+9)}} = \frac{2(s+1)}{(s+3)(s+9)-2(s+1)}$$

$$= \frac{2(s+1)}{s^2+12s+27-2s-2} = \frac{2(s+1)}{s^2+10s+25} = \frac{2(s+1)}{(s+5)^2}$$

Alternatively, the zero could be in the right half plane. Then

$$H(s) = \frac{2(s-1)}{(s+3)(s+9)}$$

and

$$G = \frac{H}{1-H} = \frac{\frac{2(s-1)}{(s+3)(s+9)}}{1-\frac{2(s-1)}{(s+3)(s+9)}} = \frac{2(s-1)}{(s+3)(s+9)-2(s-1)}$$
$$= \frac{2(s-1)}{s^2+12s+27-2s+2} = \frac{2(s-1)}{s^2+10s+29} = \frac{2(s-1)}{(s+5+j2)(s+5-j2)}$$

For a third alternative, we could negate $H(s)$ so that

$$H(s) = -\frac{2(s+1)}{(s+3)(s+9)}\,.$$

Then

$$G = \frac{H}{1-H} = \frac{-\frac{2(s+1)}{(s+3)(s+9)}}{1+\frac{2(s+1)}{(s+3)(s+9)}} = \frac{-2(s+1)}{(s+3)(s+9)+2(s+1)}$$
$$= \frac{-2(s+1)}{s^2+12s+27+2s+2} = \frac{-2(s+1)}{s^2+14s+29}$$

And finally, the zero could be in the right half plane with overall negation. Then

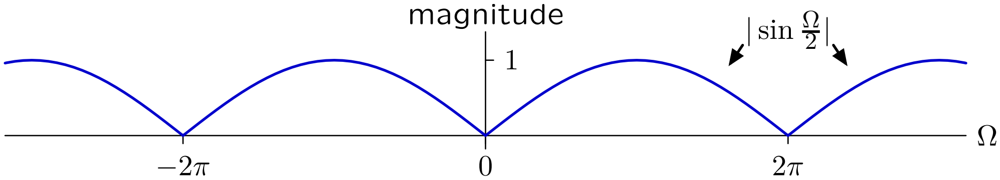
$$H(s) = -\frac{2(s-1)}{(s+3)(s+9)}$$

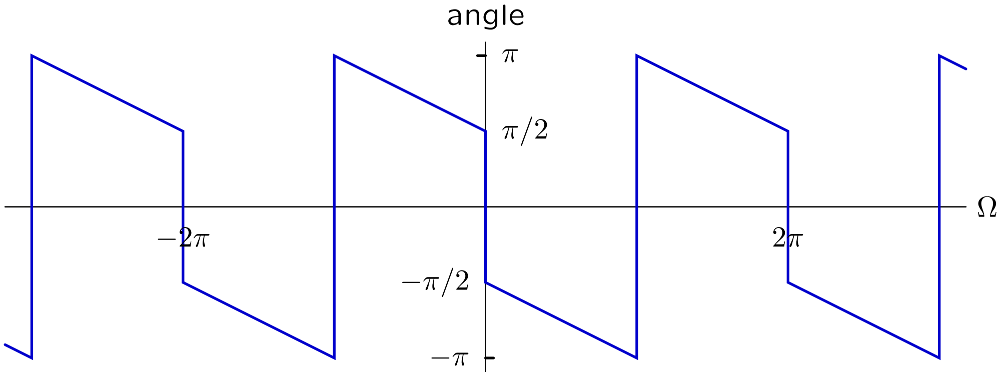
and

$$G = \frac{H}{1-H} = \frac{-\frac{2(s-1)}{(s+3)(s+9)}}{1+\frac{2(s-1)}{(s+3)(s+9)}} = \frac{-2(s-1)}{(s+3)(s+9)+2(s-1)}$$
$$= \frac{-2(s-1)}{s^2+12s+27+2s-2} = \frac{-2(s-1)}{s^2+14s+25}$$

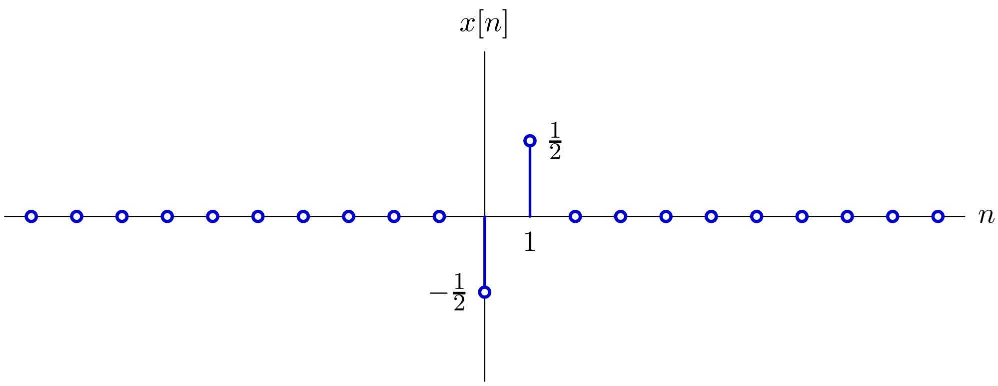
5. Inverse Fourier *[15 points]*

The magnitude and angle of the Fourier transform of $x[n]$ are shown below.





Sketch and fully label $x[n]$ on the axes below.



$$X(e^{j\Omega}) = -j\,\sin\frac{\Omega}{2}\,e^{-j\Omega/2} = -j\left(\frac{e^{j\Omega/2}-e^{-j\Omega/2}}{j2}\right)e^{-j\Omega/2} = \frac{1}{2}e^{-j\Omega} - \frac{1}{2}$$

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\Omega})e^{j\Omega n}d\Omega = \frac{1}{4\pi}\int_{2\pi}\left(e^{-j\Omega}-1\right)e^{j\Omega n}d\Omega = \frac{1}{4\pi}\int_{2\pi}\left(e^{j\Omega(n-1)}-e^{j\Omega n}\right)d\Omega$$

$$= \begin{cases} -1/2 & n=0 \\ 1/2 & n=1 \\ 0 & \text{otherwise} \end{cases}$$

Worksheet (intentionally blank)

6. Taylor series *[15 points]*

Let $x_c(t)$ represent the convolution of $x_a(t)$ with $x_b(t)$ where

$$x_a(t) = \begin{cases} a_0 + a_1 t + \frac{a_2}{2!}\,t^2 + \frac{a_3}{3!}\,t^3 + \cdots + \frac{a_n}{n!}\,t^n + \cdots & ;\ t>0 \\ 0 & ;\ \text{otherwise} \end{cases}$$

$$x_b(t) = \begin{cases} b_0 + b_1 t + \frac{b_2}{2!}\,t^2 + \frac{b_3}{3!}\,t^3 + \cdots + \frac{b_n}{n!}\,t^n + \cdots & ;\ t>0 \\ 0 & ;\ \text{otherwise} \end{cases}$$

$$x_c(t) = \begin{cases} c_0 + c_1 t + \frac{c_2}{2!}\,t^2 + \frac{c_3}{3!}\,t^3 + \cdots + \frac{c_n}{n!}\,t^n + \cdots & ;\ t>0 \\ 0 & ;\ \text{otherwise} \end{cases}$$

Determine expressions for the c coefficients (c_0, c_1, ... c_n) in terms of the a coefficients (a_0, a_1, ... a_n) and b coefficients (b_0, b_1, ... b_n).

$c_0 = \boxed{0}$

$c_1 = \boxed{a_0 b_0}$

$c_2 = \boxed{a_1 b_0 + a_0 b_1}$

$c_3 = \boxed{a_2 b_0 + a_1 b_1 + a_0 b_2}$

$c_n = \boxed{\sum_{k=1}^{n} a_{n-k} b_{k-1}}$

$$x_c = x_a * x_b = \left(\sum_{n=0}^{\infty} \frac{a_n}{n!}\,t^n\,u(t)\right) * \left(\sum_{k=0}^{\infty} \frac{b_k}{k!}\,t^k\,u(t)\right) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left(\frac{a_n}{n!}\,t^n\,u(t)\right) * \left(\frac{b_k}{k!}\,t^k\,u(t)\right)$$

$$X_c(s) = \sum_{n=0}^{\infty} \mathcal{L}\left(\frac{c_n}{n!}t^n\,u(t)\right) = \sum_{n=0}^{\infty} \frac{c_n}{s^{n+1}}$$

$$= X_a(s)X_b(s) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \mathcal{L}\left(\frac{a_n}{n!}\,t^n\,u(t)\right) \times \mathcal{L}\left(\frac{b_k}{k!}\,t^k\,u(t)\right) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{a_n b_k}{s^{n+k+2}}$$

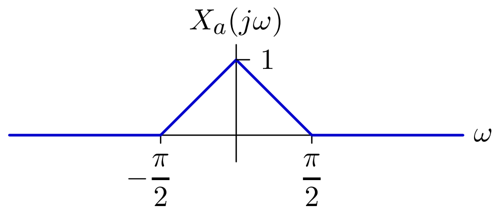
Equate terms with equal powers of s:

$$c_n = \sum_{k=1}^{n} a_{n-k} b_{k-1}$$

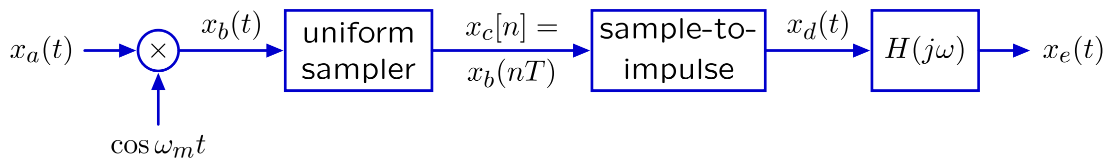
Worksheet (intentionally blank)

7. Modulated Sampling *[15 points]*

The Fourier transform of a signal $x_a(t)$ is given below.



This signal passes through the following system



where $x_d(t) = \displaystyle\sum_{n=-\infty}^{\infty} x_c[n]\delta(t - nT)$ and

$$H(j\omega) = \begin{cases} T & \text{if } |\omega| < \frac{\pi}{T} \\ 0 & \text{otherwise}. \end{cases}$$

The constants ω_m and T can be different in **Parts a-c** below.

Part a. Which (if any) of the waveforms on the following page could represent $x_b(t)$?

A-F or **none** C

If your answer was **A-F** then specify the corresponding frequency ω_m.

$\omega_m =$ 3π

If your answer was **none** then briefly explain why.

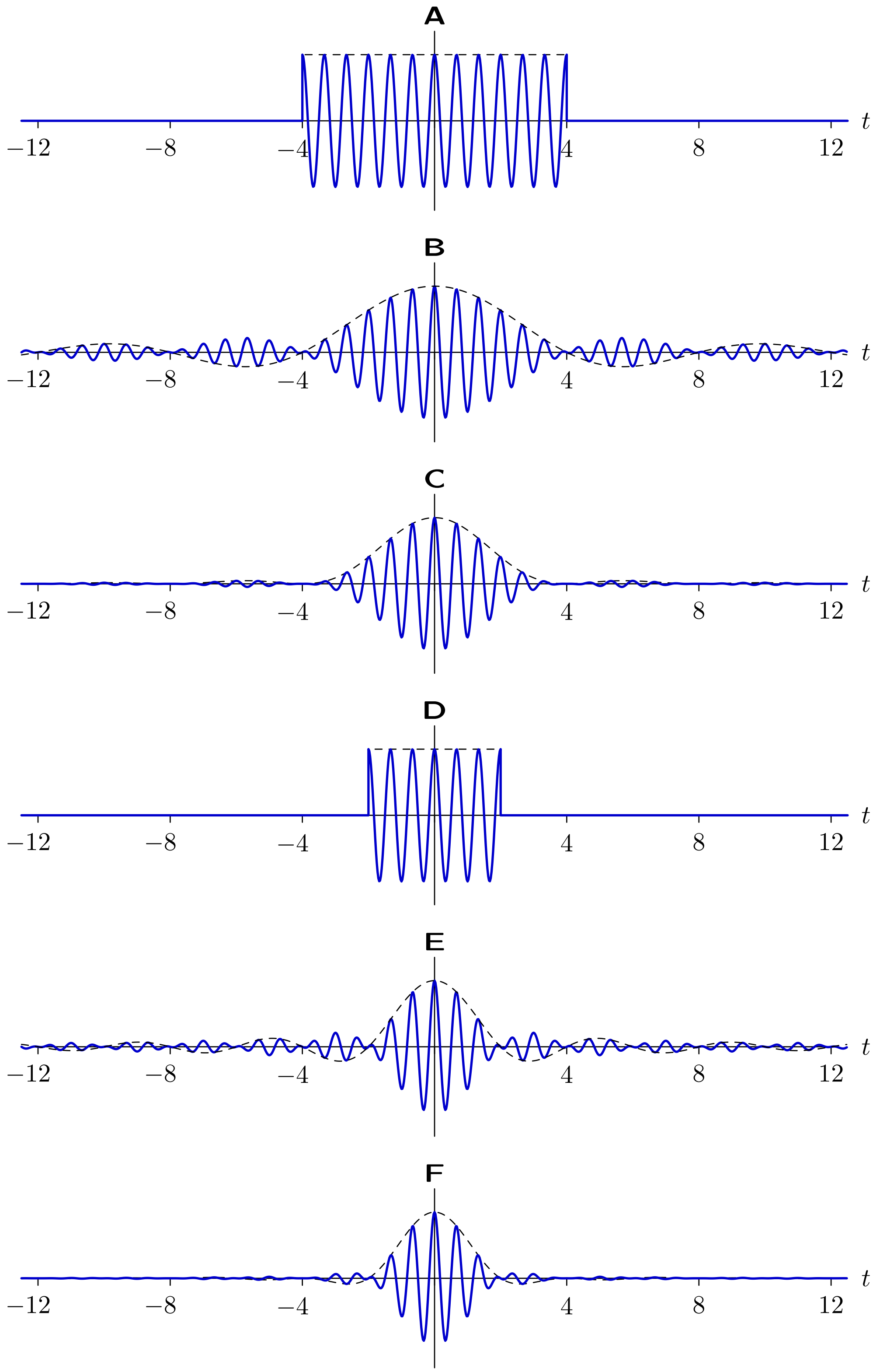
The Fourier transform $X_a(j\omega)$ can be written as the convolution of two rectangular pulses, each with height of 2 for $|\omega| < \frac{\pi}{4}$. Thus

$$x_a(t) = 4\left(\frac{\sin \frac{\pi t}{4}}{\pi t}\right)^2$$

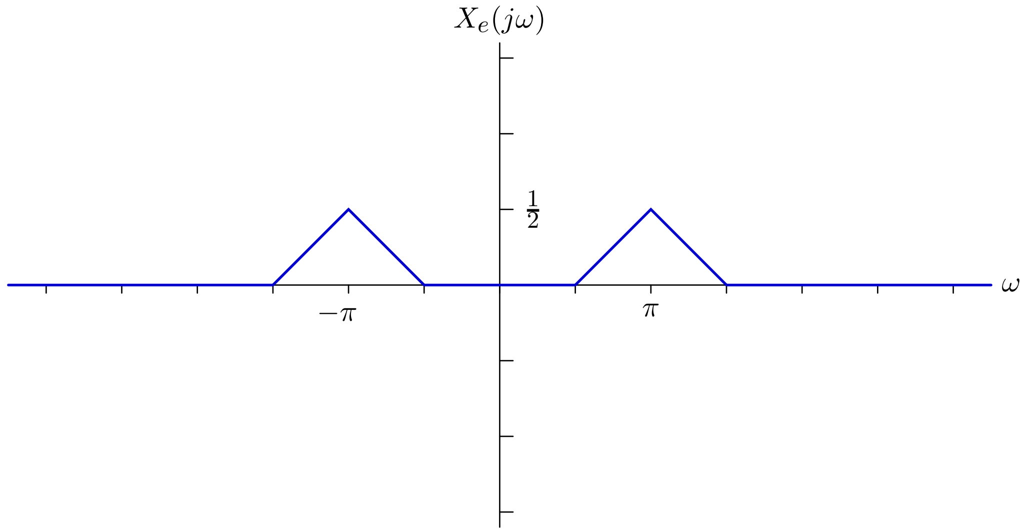
Modulation then produces

$$x_b(t) = 4\left(\frac{\sin \frac{\pi t}{4}}{\pi t}\right)^2 \cos(\omega_m t)$$

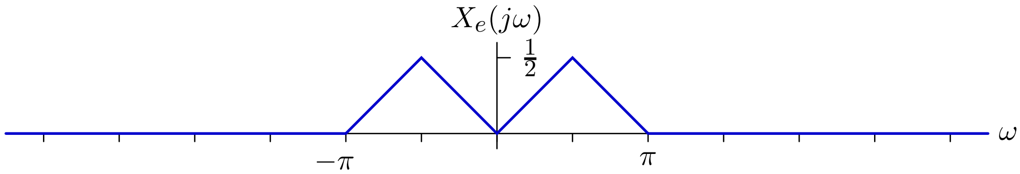
which corresponds to waveform **C** with $\omega_m = 3\pi$.


A
B
C
D
E
F
t
−12
−8
−4
4
8
12

Part b. Sketch the Fourier transform of $x_e(t)$ for the case when $\omega_m = 5\pi$ and $T = \frac{1}{3}$. Label the important features of your plot.



Part c. Is it possible to adjust T so that $X_e(j\omega)$ has the following form when $\omega_m = 5\pi$?



yes or **no**: yes

If yes, determine such a value of T (there may be multiple solutions, you need only specify one of them).

$T =$ $\frac{4}{11}$ or $\frac{4}{9}$ or $\frac{8}{9}$ or $\frac{4}{3}$

If no, briefly explain why not.

Worksheet (intentionally blank)

Worksheet (intentionally blank)

Worksheet (intentionally blank)