6.003 (Fall 2009)

Final Examination *December 17, 2009*

Name:

Kerberos Username:

Please circle your section number:

Section	*Instructor*	*Time*
1	Marc Baldo	10 am
2	Marc Baldo	11 am
3	Elfar Adalsteinsson	1 pm
4	Elfar Adalsteinsson	2 pm

Partial credit will be given for answers that demonstrate some but not all of the important conceptual issues.

Explanations are not required and will not affect your grade.

You have **three hours.**

Please put your initials on all subsequent sheets.

Enter your answers in the boxes.

This quiz is closed book, but you may use four 8.5×11 sheets of paper (eight sides total).

No calculators, computers, cell phones, music players, or other aids.

1	/15
2	/10
3	/14
4	/14
5	/15
6	/16
7	/16
Total	/100

1. CT System with Feedback *[15 points]*

Let G represent a causal system that is described by the following differential equation

$$\frac{dy(t)}{dt} + y(t) = \frac{dx(t)}{dt} - x(t)$$

where $x(t)$ represents the input signal and $y(t)$ represents the output signal.

a. Determine the output $y_1(t)$ of G when the input is

$$x_1(t) = \begin{cases} e^{-t}\,; & t \ge 0 \\ 0\,; & \text{otherwise} \end{cases}$$

Enter your result in the box below.

$$y_1(t) = \boxed{(1-2t)\,e^{-t}u(t)}$$

$$(s+1)Y = (s-1)X$$

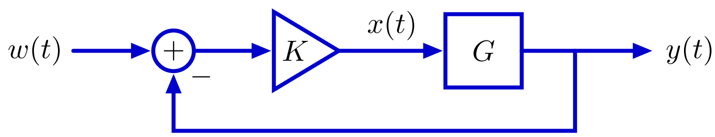
$$Y = \frac{s-1}{s+1}X$$

$$Y_1 = \frac{s-1}{s+1}X_1\,; \quad X_1 = \frac{1}{s+1}\,; \quad \operatorname{Re} s > -1$$

$$Y_1 = \frac{s-1}{(s+1)^2} = \frac{1}{s+1} - \frac{2}{(s+1)^2}$$

$$y_1(t) = e^{-t}u(t) - 2t\,e^{-t}u(t)$$

Now consider a feedback loop that contains the G system described on the previous page.[1]



b. Determine a differential equation that relates $w(t)$ to $y(t)$ when $K = 10$. The differential equation should not contain references to $x(t)$.

Enter the differential equation in the box below.

$$11\frac{dy(t)}{dt} - 9y(t) = 10\frac{dw(t)}{dt} - 10w(t)$$

$$\frac{Y}{W} = \frac{K\,\dfrac{s-1}{s+1}}{1+K\,\dfrac{s-1}{s+1}} = K\,\frac{s-1}{s+1+Ks-K} = K\,\frac{s-1}{(K+1)s-(K-1)} = K\,\frac{s-1}{11s-9}$$

[1] The minus sign near the adder indicates that the output of the adder is $w(t) - y(t)$

c. Determine the values of K for which the feedback system on the previous page is stable. Enter the range (or ranges) in the box below.

$$-1 < K < 1$$

$$\frac{Y}{W} = \frac{s-1}{(K+1)s-(K-1)}$$

$$s = \frac{K-1}{K+1} < 0$$

To be stable, the pole must be in the left half-plane.

Consider two cases: either K is greater than -1 or it is less than -1.

If $K > -1$, then the denominator is positive and $K-1$ must be less than 0 to assure that the pole is in the left half-plane. This implies $K < 1$. Thus K could be in the range for -1 to 1.

Now consider the possibility that K is less than -1. Then $K-1$ would have to be greater than 0 for the pole to be in the left half-plane. But this could only happen if $K > 1$, which contradicts the assumption that $K < -1$. Therefore K cannot be less than -1.

2. Stepping Up and Down *[10 points]*

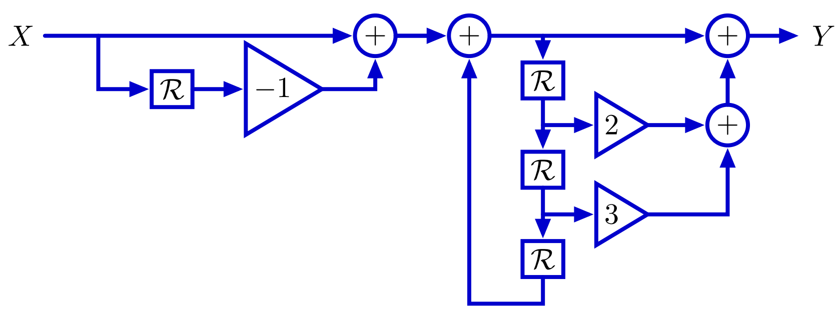
Use a small number of delays, gains, and 2-input adders (and no other types of elements) to implement a system whose response (starting at rest) to a **unit-step** signal

$$x[n] = u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

is

$$y[n] = \begin{cases} 0 & n < 0 \\ 1 & n = 0, 3, 6, 9, \ldots \\ 2 & n = 1, 4, 7, 10, \ldots \\ 3 & n = 2, 5, 8, 11, \ldots \end{cases}$$

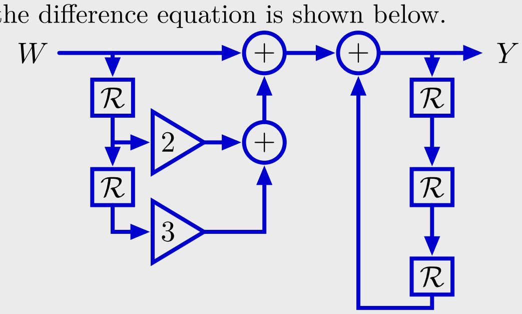
Draw a block diagram of your system below.



First find a system whose unit-sample response is the desired sequence. The periodicity of 3 suggests that $y[n]$ depends on $y[n-3]$. To get the correct numbers, just delay the input and weight the delays appropriately. The resulting difference equation is

$$y[n] = y[n-3] + w[n] + 2w[n-1] + 3w[n-2]\,.$$

A direct realization of the difference equation is shown below.



We can "reuse" 2 delays by commuting the left and right parts of this network.

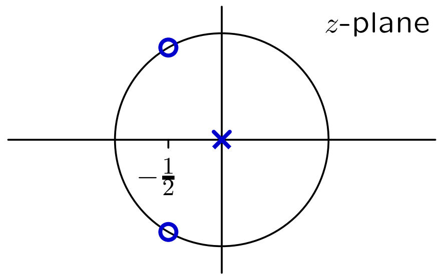
Next compute $w[n]$ which is the first difference of $x[n]$:

$$w[n] = x[n] - x[n-1]$$

The result is the cascade of the first difference and the previous result.

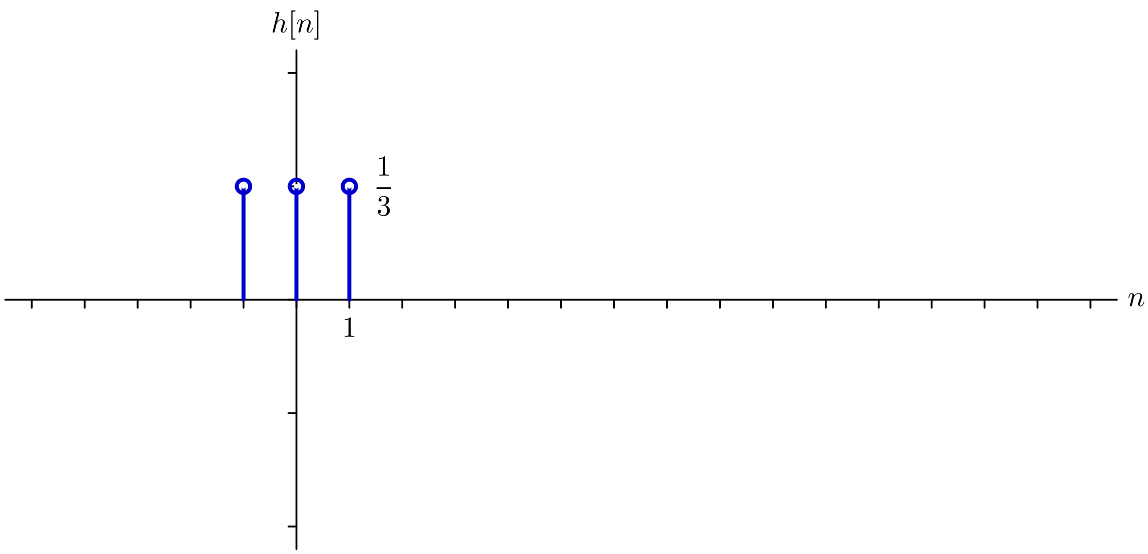
3. DT systems *[14 points]*

The pole-zero diagram for a DT system is shown below, where the circle has radius 1.



It is known that when the input is 1 for all n, the output is also 1 for all n.

Sketch the unit-sample response $h[n]$ of the system on the axes below. Label the important features of your sketch.



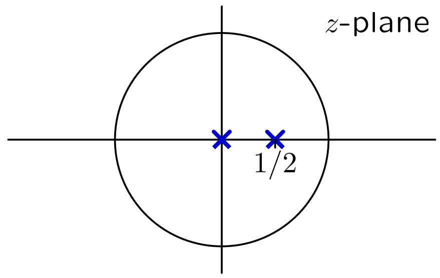
$$\frac{Y}{X} = K\,\frac{\left(z+\frac{1}{2}-j\frac{\sqrt{3}}{2}\right)\left(z+\frac{1}{2}+j\frac{\sqrt{3}}{2}\right)}{z} = K\,\frac{\left(z+\frac{1}{2}\right)^2+\frac{3}{4}}{z} = K\,\frac{z^2+z+1}{z} = K\left(z{+}1{+}z^{-1}\right)$$

$$H(1) = 3K = 1$$

Therefore, $K = \frac{1}{3}$.

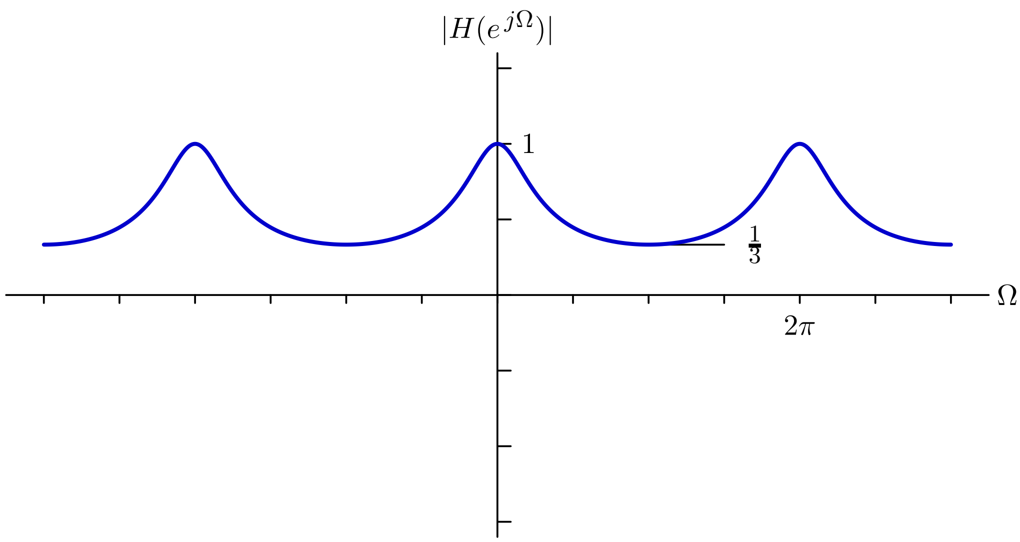
$$h[n] = \frac{1}{3}\left(\delta[n-1]+\delta[n]+\delta[n+1]\right)$$

A second DT system has the following pole-zero diagram:

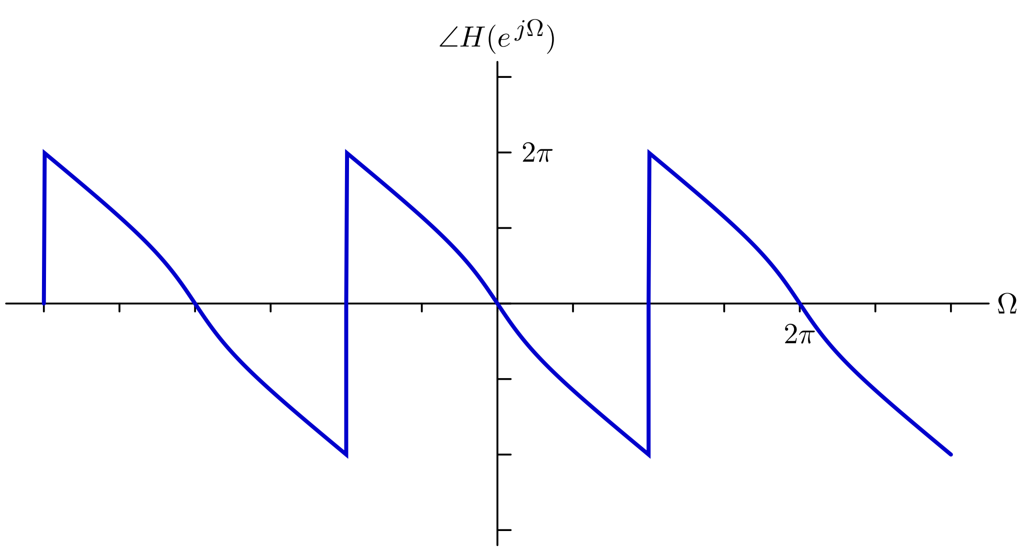


It is known that the system function $H(z)$ is 1 when $z = 1$.

Sketch the magnitude of the frequency response of this system on the axes below. Label the important features of your sketch (including the axes).

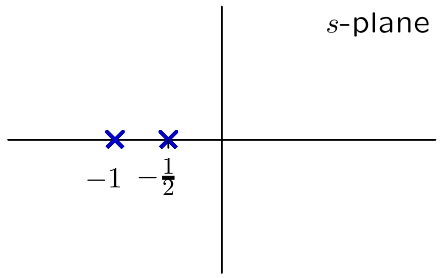


Sketch the angle of the frequency response of this system on the axes below. Label the important features of your sketch (including the axes).



4. CT Systems *[14 points]*

A causal CT system has the following pole-zero diagram:



Let $y(t) = s(t)$ represent the response of this system to a unit-step signal

$$x(t) = u(t) = \begin{cases} 1\,; & t \geq 0 \\ 0\,; & \text{otherwise}\,. \end{cases}$$

Assume that the unit-step response $s(t)$ of this system is known to approach 1 as $t \to \infty$. Determine $y(t) = s(t)$ and enter it in the box below.

$$y(t) = \boxed{(1 + e^{-t} - 2e^{-t/2})\, u(t)}$$

From the pole-zero diagram,

$$H(s) = \frac{K}{(s+1)(s+\frac{1}{2})}\,.$$

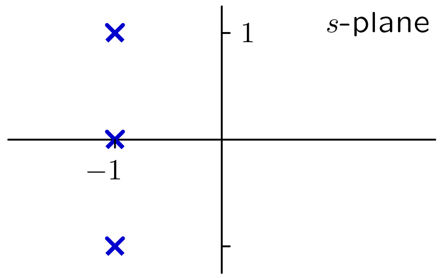
Since the system is stable (system is causal and poles are all in left half-plane), the unit-step response will approach $H(0)$ as $t \to \infty$. Therefore

$$H(0) = \frac{K}{(1)(\frac{1}{2})} = 2K = 1$$

and $K = \frac{1}{2}$. The Laplace transform of the unit step is $X(s) = \dfrac{1}{s}$ for $\text{Re}\{s\} > 0$. Therefore

$$Y = \frac{\frac{1}{2}}{s(s+1)(s+\frac{1}{2})} = \frac{1}{s} + \frac{1}{s+1} - \frac{2}{s+\frac{1}{2}}$$

A second CT system has the following pole-zero diagram:



Assume that the input signal is

$$x(t)=\begin{cases}1\,; & \cos t>\frac{1}{\pi}\\ 0\,; & \text{otherwise}\,.\end{cases}$$

Let a_k and b_k represent the Fourier series coefficients of the input and output signals, respectively, where the fundamental (lowest frequency component) of each signal has a period of 2π.

It is known that $\dfrac{b_0}{a_0}=1$. Determine $\dfrac{b_1}{a_1}$.

$$\frac{b_1}{a_1}=\boxed{\frac{2}{-1+3j}}$$

$$H(s)=\frac{K}{(s+1)(s+1-j\omega)(s+1+j\omega)}=\frac{K}{(s+1)((s+1)^2+1)}=\frac{K}{(s+1)(s^2+2s+2)}$$

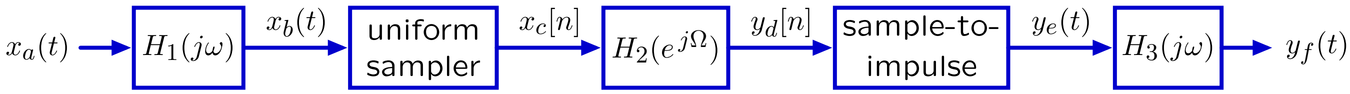
$$\frac{b_0}{a_0}=1=H(j0)=\frac{K}{2}$$

The fundamental frequency is $\omega=\frac{2\pi}{2\pi}=1$.

$$\frac{b_1}{a_1}=H(j1)=\frac{K}{(1+j)(1+2j)}=\frac{2}{-1+3j}$$

5. DT processing of CT signals *[15 points]*

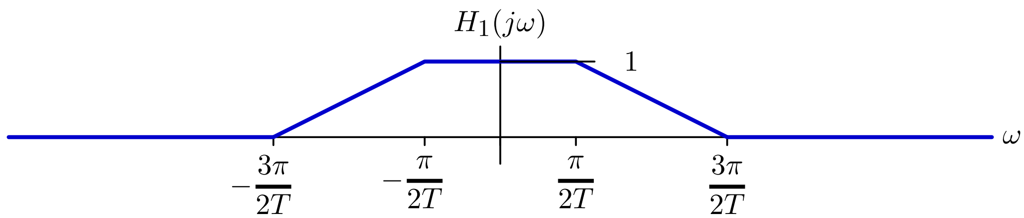
Consider the following system for DT processing of CT signals:

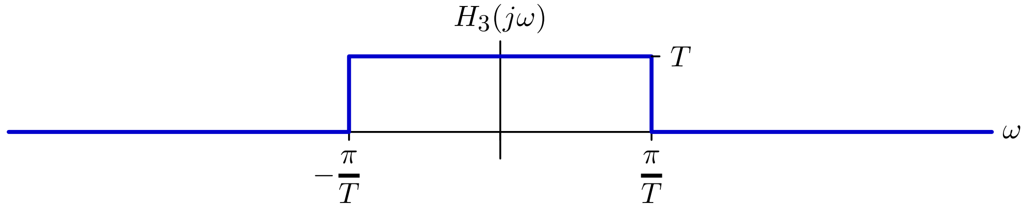


where $x_c[n] = x_b(nT)$ and

$$y_e(t) = \sum_{n=-\infty}^{\infty} y_d[n]\delta(t - nT)\,.$$

The frequency responses $H_1(j\omega)$ and $H_3(j\omega)$ are given below.





a. Assume in this part that $H_2(e^{j\Omega}) = 1$ for all frequencies Ω. Determine $y_f(t)$ when

$$x_a(t) = \cos\left(\frac{\pi}{2T}t\right) + \sin\left(\frac{5\pi}{4T}t\right)\,.$$

$y_f(t) =$ $\boxed{\cos\left(\frac{\pi}{2T}t\right) - \frac{1}{4}\sin\left(\frac{3\pi}{4T}t\right)}$

b. For this part, assume that

$$H_2(e^{j\Omega}) = \begin{cases} 1\,; & |\Omega| < \Omega_c \\ 0\,; & \Omega_c < |\Omega| < \pi\,. \end{cases}$$

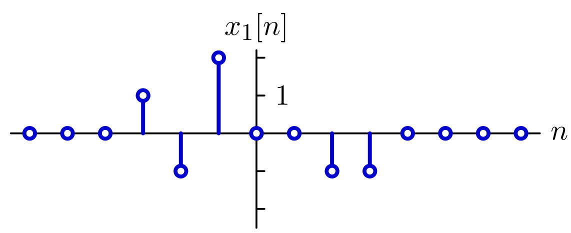
For what values of Ω_c is the overall system from $x_a(t)$ to $y_f(t)$ linear and time-invariant?

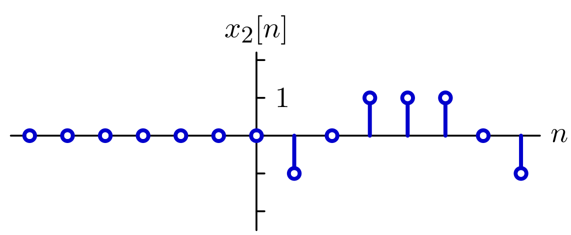
values of Ω_c: $\boxed{0 < \Omega_c < \frac{\pi}{2}}$

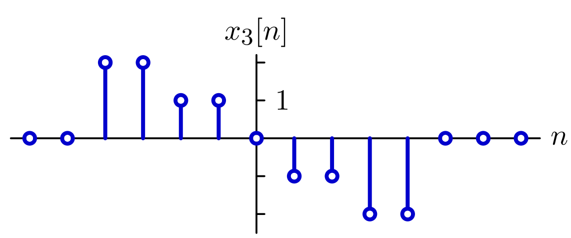
6. Which are True? *[16 points]*

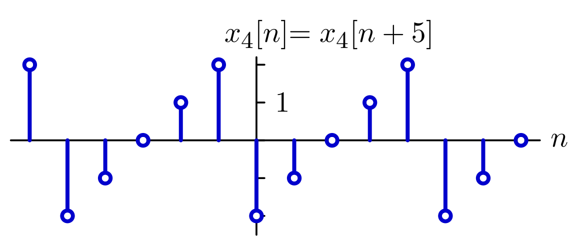
For each of the DT signals $x_1[n]$ through $x_4[n]$ (below), determine whether the conditions listed in the following table are satisfied, and answer **T** for true or **F** for false.

	$x_1[n]$	$x_2[n]$	$x_3[n]$	$x_4[n]$
$X(e^{j0}) = 0$	*T*	*F*	*T*	*T*
$\int_{-\pi}^{\pi} X(e^{j\Omega})\, d\Omega = 0$	*T*	*T*	*T*	*F*
$X(e^{j\Omega})$ is purely imaginary	*F*	*F*	*T*	*F*
$e^{jk\Omega}X(e^{j\Omega})$ is purely real for some integer k	*F*	*T*	*F*	*F*





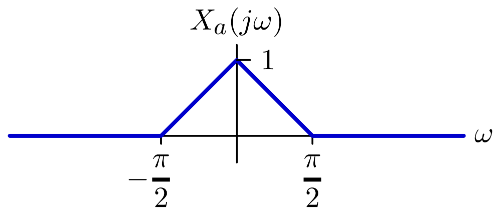




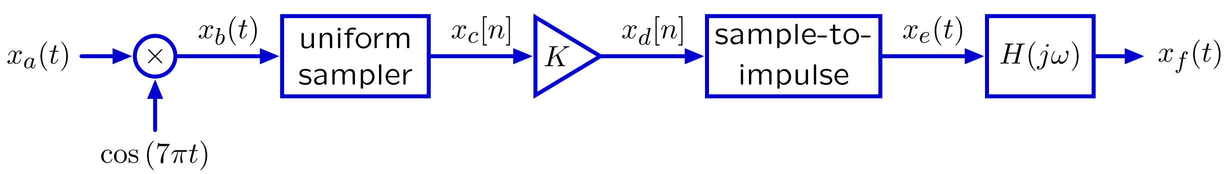
Worksheet (intentionally blank)

7. Multiplied Sampling *[16 points]*

The Fourier transform of a signal $x_a(t)$ is given below.



This signal passes through the following system



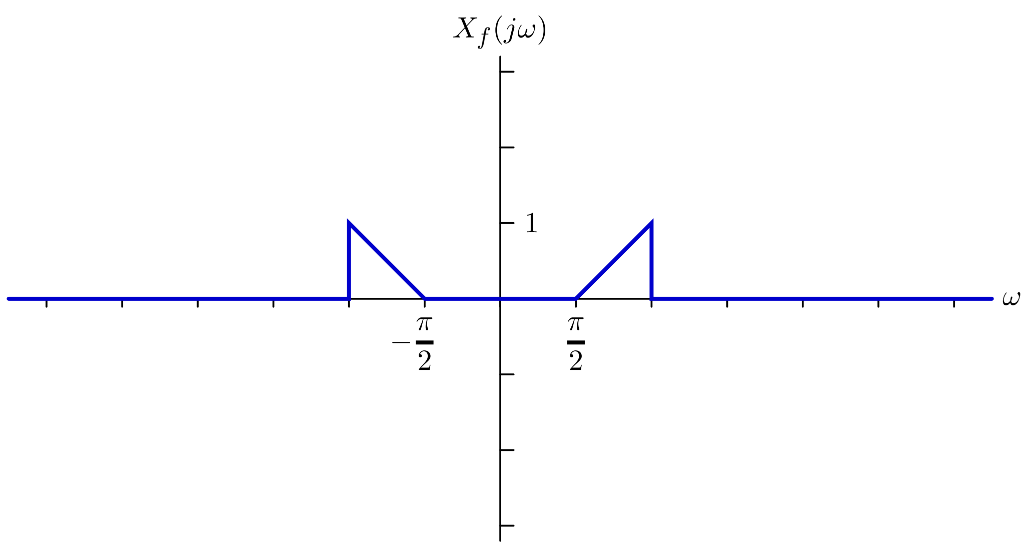
where $x_c[n] = x_b(nT)$ and

$$x_e(t) = \sum_{n=-\infty}^{\infty} x_d[n]\delta(t - nT)$$

and

$$H(j\omega) = \begin{cases} T & \text{if } |\omega| < \frac{\pi}{T} \\ 0 & \text{otherwise}. \end{cases}$$

a. Sketch the Fourier transform of $x_f(t)$ for the case when $K = 1$ and $T = 1$. Label the important features of your plot.



b. Is it possible to adjust T and K so that $x_f(t) = x_a(t)$?

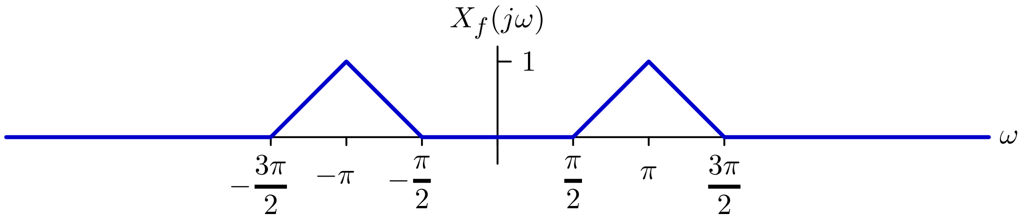
yes or **no**: yes

If yes, specify a value T and the corresponding value of K (there may be multiple solutions, you need only specify one of them).

$T =$ $\frac{2}{7}, \frac{4}{7}, \frac{6}{7}, \frac{8}{7}, \frac{10}{7}, \frac{12}{7}$, or 2 $\qquad K =$ 1

If no, briefly explain why not.

c. Is it possible to adjust T and K so that the Fourier transform of $x_f(t)$ is equal to the following, and is zero outside the indicated range?



yes or **no**: yes

If yes, specify all possible pairs of T and K that work in the table below. If there are more rows in the table than are needed, leave the remaining entries blank.

T	K
$\frac{1}{3}$	2
$\frac{2}{3}$	2
$\frac{1}{4}$	2
$\frac{1}{2}$	2

If no, briefly explain why not.

Worksheet (intentionally blank)

Worksheet (intentionally blank)

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