6.003 Homework 1

Due at the beginning of recitation on ***Wednesday, February 10, 2010****.*

Problems

1. Independent and Dependent Variables

Assume that the height of a water wave is given by $g(x - vt)$ where x is distance, v is velocity, and t is time. Assume that the height of the wave is a sinusoidal function of distance at each instant of time. Also assume that the positive peaks have a height of 1 meter (relative to the average water level) and that they occur at integer multiples of 2 meters when the time $t = 3$ seconds.

a. Determine an expression for the height of the wave $h(x, t)$ as a function of distance x and time t if the wave is traveling in the positive x direction at 5 meters/second. What is the function $g(\cdot)$ for this case?

b. Determine an expression for the height of the wave $h(x, t)$ as a function of distance x and time t if the wave is traveling in the negative x direction at 4 meters/second. What is the function $g(\cdot)$ for this case?

c. Determine the speed of the wave if successive positive peaks at $x = 1.3$ meters are separated by 0.75 seconds.

2. Even and Odd

The even and odd parts of a signal $x[n]$ are defined by the following:

- $x_e[-n] = x_e[n]$ (i.e., x_e is an even function of n)
- $x_o[-n] = -x_o[n]$ (i.e., x_o is an odd function of n)
- $x[n] = x_e[n] + x_o[n]$

Let x represent the signal whose samples are given by

$$x[n] = \begin{cases} \left(\frac{1}{2}\right)^n & n \geq 0 \\ 0 & \text{otherwise} \end{cases}.$$

a. Determine the even and odd parts of the signal x.

b. Show that your answer in part a is unique.

c. Plot the results of part a.

3. Geometric sums

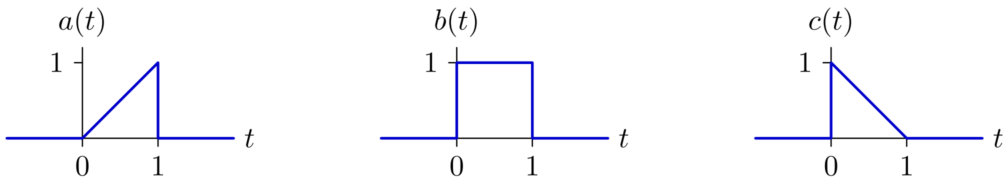
a. Expand $\dfrac{1}{1-a}$ in a power series. For what range of a does your answer converge?

b. Find a closed-form expression for $\sum_{n=0}^{N-1} a^n$. For what range of a does your answer converge?

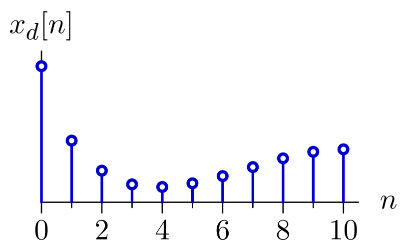
c. Expand $\dfrac{1}{(1-a)^2}$ in a power series. For what range of a does your answer converge?

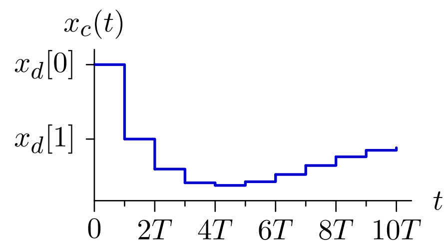
4. Reconstructing CT Signals from Samples

Let $a(t)$, $b(t)$, and $c(t)$ represent the following functions of time.



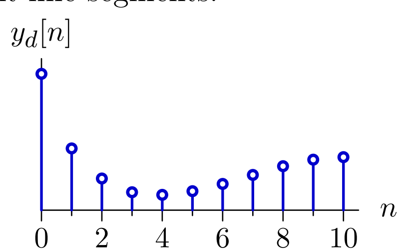
Let $x_c(t)$ represent a continuous-time signal derived from the discrete-time signal $x_d[n]$ using a zero-order hold, as illustrated below, where consecutive samples of x_d are separated by T seconds in x_c.

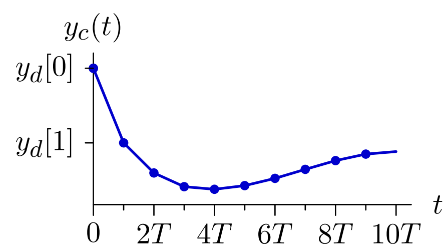




a. Determine an expression for $x_c(t)$ in terms of the samples $x_d[n]$ and the functions $a(t)$, $b(t)$, and $c(t)$. Your expression should match $x_c(t)$ at all points in time except possibly at integer multiples of T, where $x_c(t)$ is discontinuous. [Extra credit: Briefly discuss whether it is possible to develop an expression that matches $x_c(t)$ at all points, including those where $x_c(t)$ is discontinuous.]

Let $y_c(t)$ represent a continuous-time signal derived from the discrete-time signal $y_d[n]$ using a piecewise linear interpolator, so that sucessive samples of y_d are connected by straight line segments.



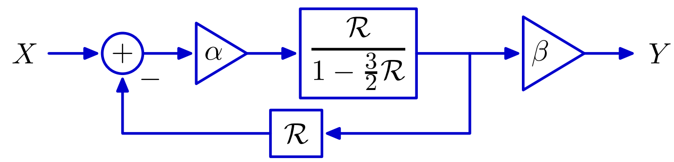


b. Determine an expression for $y_c(t)$ in terms of the samples $y_d[n]$ and the functions $a(t)$, $b(t)$, and $c(t)$. [Your expression need not match $x(t)$ at integer multiples of T.]

c. Determine an expression for $\dfrac{dy_c(t)}{dt}$ in terms of the samples $y_d[n]$ and the functions $a(t)$, $b(t)$, and $c(t)$. [Your expression need not match $x(t)$ at integer multiples of T.]

5. Missing Parameters

Consider the following system.



Assume that X is the unit-sample signal, $x[n] = \delta[n]$. Determine the values of α and β for which $y[n]$ is the following sequence (i.e., $y[0]$, $y[1]$, $y[2]$, ...):

$$0\,,\ 1\,,\ \frac{3}{2}\,,\ \frac{7}{4}\,,\ \frac{15}{8}\,,\ \frac{31}{16}\,,\ \ldots$$

Engineering Design Problems

6. Choosing a bank

Consider two banks. Bank #1 offers a 3% annual interest rate, but charges a \$1 service charge each year, including the year when the account was opened. Bank #2 offers a 2% annual interest rate, and has no annual service charge. Let $y_i[n]$ represent the balance in bank i at the beginning of year n and $x_i[n]$ represent the amount of money you deposit in bank i during year n. Assume that deposits during year n are credited to the balance at the end of that year but earn no interest until the following year.

a. Use difference equations to express the relation between deposits and balances for each bank.

b. Assume that you deposit \$100 in each bank and make no further deposits. Solve your difference equations in part a numerically (using Matlab, Octave, or Python) to determine your balance in each bank for the next 25 years. Make a plot of these balances. Which account has the larger balance 5 years after the initial investment (one year without interest and 4 years with interest). Which account has the larger balance after 25 years (i.e., at the beginning of the 26$^{\text{th}}$ year) [Hint: See the Appendix for help with programming.]

7. Drug dosing

When drugs are used to treat a medical condition, doctors often recommend starting with a higher dose on the first day than on subsequent days. In this problem, we consider a simple model to understand why. Assume that the human body is a tank of blood and that drugs instantly dissolve in the blood when ingested. Further assume that drug vanishes from the blood (either because it is broken down or because it is flushed by the kidneys) at a rate that is proportional to drug concentration.

Let $x[n]$ represent the amount of drug taken on day n, and let $y[n]$ represent the total amount of drug in the blood on day n, just after the dose $x[n]$ has dissolved in the blood, so that

$$y[n] = x[n] + \alpha y[n-1]\,.$$

a. Determine the amount of drug in the blood that would result after taking one unit of drug each day for many consecutive days, i.e., determine $\lim_{n\to\infty} y[n]$ when $x[n] = 1$.

b. Assume that there is initially no drug in the blood. Then, starting on day 0, one unit of drug is taken each day. Determine the first day when the amount of drug in the blood will equal or exceed half of its final value.

c. Consider the following table of doses and resulting amount of drug in the blood:

n	$x[n]$	$y[n]$
-1	0	0
0	1	1.00
1	1	1.73
2	1	2.27
3	1	2.66
4	1	2.95
5	1	3.16
6	1	3.32
7	1	3.43

Notice that the blood concentration ramps up over the first few days. Suggest a different initial dose $x[0]$ that will result in a more constant amount of drug in the blood (with $x[n]$ remaining at 1 for all $n \geq 1$). Make a table to show your result.

Appendix: Fibonacci code

You may use Python and/or Matlab/Octave to solve problems in this homework assignment. Octave is a free-software linear-algebra solver, with a syntax that is similar to that of Matlab. Octave is available for most platforms. See **www.octave.org**.

The following code calculates, prints, and plots the first 20 Fibonacci numbers (i.e., $f[0]$ through $f[19]$).

Example Matlab/Octave code

```
y(1) = 1;                     % initial conditions
y(2) = 1;                     % indices start at 1 (not 0)
for i = 3:20
    y(i) = y(i-1)+y(i-2)
end
y                             % print y
stem(0:19,y)
```

Example Python code

```
from pylab import *
y = [1,1]                     # initial conditions
for i in range(2,20):
    y.append(y[i-1]+y[i-2])
print y
stem(range(20),y)
show()
```