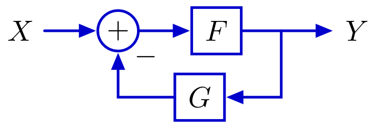
6.003 Homework 2

*Due at the beginning of recitation on **Wednesday, February 17, 2010**.*

Problems

1. Black's Equation

Consider the general form of a feedback problem:



Notice the minus sign on the adder: it indicates that the lower input is subtracted rather than added.

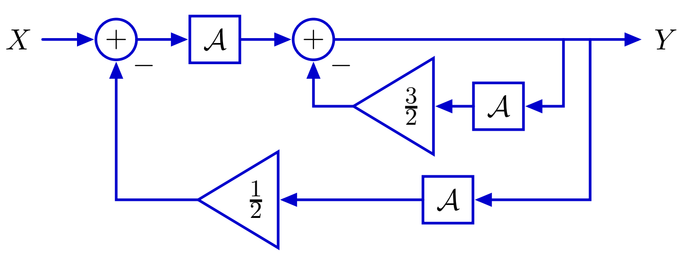
a. Determine the system functional $H = \frac{Y}{X}$. This result is known as Black's equation.

b. Assume that F can be written as a ratio of polynomials in $\mathcal{R}$. Let N_F and D_F represent the numerator and denominator polynomial, respectively. Similarly, assume that G can be written as a ratio of polynomials N_G and D_G. Express the system functional $\frac{Y}{X}$ in terms of N_F, D_F, N_G, and D_G. Can $\frac{Y}{X}$ be expressed as a ratio of polynomials in $\mathcal{R}$?

c. Assume that F can be written as a ratio of polynomials in $\mathcal{A}$. Let N_F and D_F represent the numerator and denominator polynomial, respectively. Similarly, assume that G can be written as a ratio of polynomials N_G and D_G. Express the system functional $\frac{Y}{X}$ in terms of N_F, D_F, N_G, and D_G. Can $\frac{Y}{X}$ be expressed as a ratio of polynomials in $\mathcal{A}$?

2. Characterizing block diagrams

Consider the system defined by the following block diagram:



a. Determine the system functional $H = \frac{Y}{X}$.

b. Determine the poles of the system.

c. Determine the impulse response of the system.

3. Finding a system

a. Determine the difference equation and block diagram representations for a system whose output is $10, 1, 1, 1, 1, \ldots$ when the input is $1, 1, 1, 1, 1, \ldots$.

b. Determine the difference equation and block diagram representations for a system whose output is $1, 1, 1, 1, 1, \ldots$ when the input is $10, 1, 1, 1, 1, \ldots$.

c. Compare the difference equations in parts a and b. Compare the block diagrams in parts a and b.

4. Scaling time

A system containing only adders, gains, and delays was designed with system functional

$$H = \frac{Y}{X}$$

which is a ratio of two polynomials in $\mathcal{R}$. When this system was constructed, users were dissatisfied with its responses. Engineers then designed three new systems, each based on a different idea for how to modify H to improve the responses.

System H_1: every delay element in H is replaced by a cascade of two delay elements.

System H_2: every delay element in H is replaced by a gain of $\frac{1}{2}$ followed by a delay.

System H_3: every delay element in H is replaced by a cascade of three delay elements.

For each of the following parts, evaluate the truth of the associated statement and enter **yes** if the statement is always true or **no** otherwise.

a. If H has a pole at $z = j = \sqrt{-1}$, then H_1 has a pole at $z = e^{j5\pi/4}$.

b. If H has a pole at $z = p$ then H_2 has a pole at $z = 2p$.

c. If H is stable then H_3 is also stable (where a system is said to be stable if all of its poles are inside the unit circle).

Engineering Design Problems

5. Repeated Poles

Consider a system H whose unit-sample response is

$$h[n] = \begin{cases} n+1 & \text{for } n \geq 0 \\ 0 & \text{otherwise}. \end{cases}$$

a. Determine the poles of H.

b. H can be written as the cascade of two identical subsystems, each called G. Determine the difference equation for G.

c. Draw a block diagram for H using just adders, gains, and delays. Use the block diagram to explain why the unit-sample response of H is the sequence $h[n] = n + 1$, $n \geq 0$.

d. Because the system functional has two poles at the same location, the unit-sample response of H cannot be expressed as a weighted sum of geometric sequences,

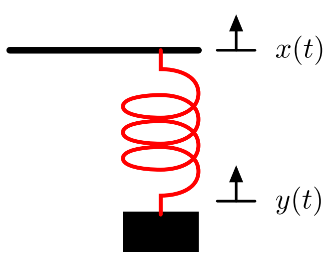
$$h[n] = a_0 z_0^n + a_1 z_1^n \,; \quad n \geq 0\,.$$

However, h can be written in the previous form if the poles of H are displaced from their true positions by a small amounts (e.g., one pole by $+\epsilon$ and the other by $-\epsilon$). Determine a_0, a_1, z_0, and z_1 as functions of ϵ.

e. Compare the results of the approximation in part c for different values of ϵ.

6. Masses and Springs, Forwards and Backwards

The following figure illustrates a mass and spring system. The input $x(t)$ represents the position of the top of the spring. The output $y(t)$ represents the position of the mass.



The mass is $M = 1\,\text{kg}$ and the spring constant is $K = 1\,\text{N/m}$. Assume that the spring obeys Hooke's law and that the reference positions are defined so that if the input $x(t)$ is equal to zero, then the resting position of $y(t)$ is also zero.

a. Determine a differential equation that relates the input $x(t)$ and output $y(t)$.

b. Calculate the step response of the system.

c. The differential equation in part a contains a second derivative (if you did part a correctly). We wish to develop a forward-Euler approximation for this derivatve. One method is to write the second-order differential equation in part a as a part of first order differential equations. Then apply the forward-Eular approximation to the first order derivatives:

$$\left.\frac{dy(t)}{dt}\right|_{t=nT} \approx \frac{y[n+1]-y[n]}{T}\,.$$

Use this approach to find a difference equation to approximate the behavior of the mass and spring system. Calculate the step response of the system and compare your results to those in part b.

d. An alternative to the forward-Euler approximation is the backward-Euler approximation:

$$\left.\frac{dy(t)}{dt}\right|_{t=nT} \approx \frac{y[n]-y[n-1]}{T}\,.$$

Repeat the exercise in the previous part, but using the backward-Euler approximation instead of the forward-Euler approximation.

e. The forward-Euler method approximates the second derivative at $t = nT$ as

$$\left.\frac{d^2y(t)}{dt^2}\right|_{t=nT} = \frac{y[n+2]-2y[n+1]+y[n]}{T^2}\,.$$

The backward-Euler method approximates the second derivative at $t = nT$ as

$$\left.\frac{d^2y(t)}{dt^2}\right|_{t=nT} = \frac{y[n]-2y[n-1]+y[n-2]}{T^2}\,.$$

Consider a compromise based on a centered approximation:

$$\left.\frac{d^2y(t)}{dt^2}\right|_{t=nT} = \frac{y[n+1]-2y[n]+y[n-1]}{T^2}\,.$$

Use this approximation to determine the step response of the system. Compare your results to those in the two previous parts of this problem.