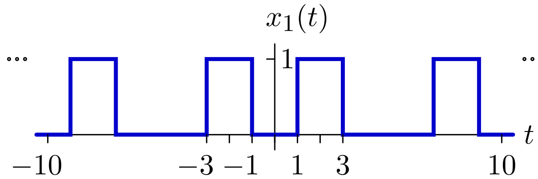
6.003 Homework 9

Due at the beginning of recitation on ***Wednesday, April 14, 2010***.

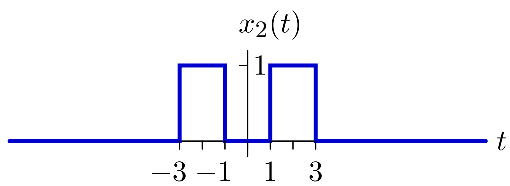
Problems

1. Fourier varieties

a. Determine the Fourier series coefficients of the following signal, which is periodic in $T = 10$.

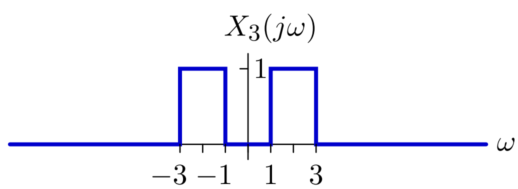


b. Determine the Fourier transform of the following signal, which is zero outside the indicated range.



What is the relation between the answer to this part and that of the previous part?

c. Determine the time waveform that corresponds to the following Fourier transform, which is zero outside the indicated range.



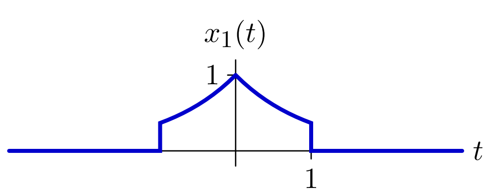
What is the relation between the answer to this part and that of the previous part?

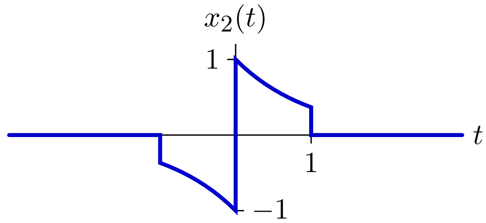
2. Fourier transform properties

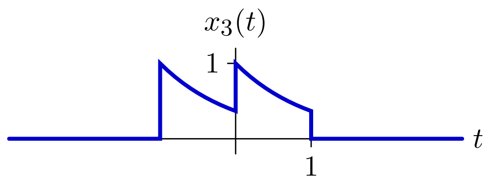
Let $X(j\omega)$ represent the Fourier transform of

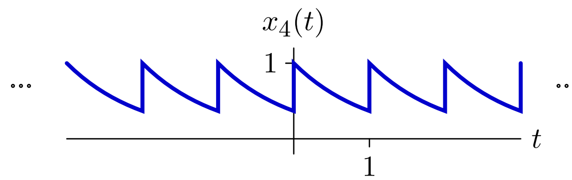
$$x(t) = \begin{cases} e^{-t} & 0 < t < 1 \\ 0 & \text{otherwise} \end{cases}$$

Express the Fourier Transforms of each of the following signals in terms of $X(j\omega)$.









3. Fourier transforms

Find the Fourier transforms of the following signals.

a. $x_1(t) = e^{-|t|}\cos(2t)$

b. $x_2(t) = \dfrac{\sin(2\pi t)}{\pi(t-1)}$

c. $x_3(t) = \begin{cases} t^2 & 0 < t < 1 \\ 0 & \text{otherwise} \end{cases}$

d. $x_4(t) = (1 - |t|)\, u(t+1)u(1-t)$

4. Parseval's theorem

Parseval's theorem relates time- and frequency-domain methods for calculating the average energy of a signal as follows:

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

where a_k represents the Fourier series coefficients of the periodic signal $x(t)$ with period T.

a. We can derive Parseval's theorem from the properties of CT Fourier series.

1. Let $y(t) = |x(t)|^2$. Find the Fourier series coefficients b_k of $y(t)$.
[Hint: $|x(t)|^2 = x(t)x^*(t)$.]

2. Use the result from the previous part to derive Parseval's theorem.

b. Let $x_1(t)$ represent the input to an LTI system, where

$$x_1(t) = \sum_{k=-\infty}^{\infty} \alpha^{|k|} e^{jk\frac{\pi}{4}t}$$

for $0 < \alpha < 1$. The frequency response of the system is

$$H(j\omega) = \begin{cases} 1 & |\omega| < W \\ 0 & \text{otherwise.} \end{cases}$$

What is the minimum value of W so that the average energy in the output signal will be at least 90% of that in the input signal.

Engineering Design Problem

5. Overshoot

a. What function $f(t)$ has the Fourier series

$$\sum_{n=1}^{\infty} \frac{\sin nt}{n}?$$

You can evaluate the sum analytically or numerically. Either way, guess a closed form for $f(t)$ and then sketch it.

b. Confirm your conjecture for $f(t)$ by finding the Fourier series coefficients f_n for $f(t)$. Compare your result to the expression in the previous part. What happens to the cosine terms?

c. Define the partial sum

$$f_N(t) = \sum_{n=1}^{N} \frac{\sin nt}{n},$$

Plot some $f_N(t)$'s. By what fraction does $f_N(t)$ overshoot $f(t)$ at worst? Does that fraction tend to zero or to a finite value as $N \to \infty$? If it is a finite value, estimate it.

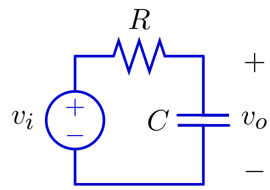
d. Now define the average of the partial sums:

$$F_N(t) = \frac{f_1(t) + f_2(t) + f_3(t) + \cdots + f_N(t)}{N}$$

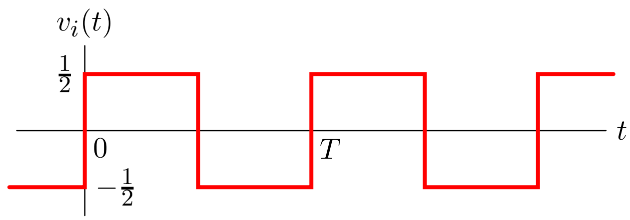
Plot some $F_N(t)$'s. Compare your plots with those of $f_N(t)$ that you made in the previous part, and qualitatively explain any differences.

6. Filtering

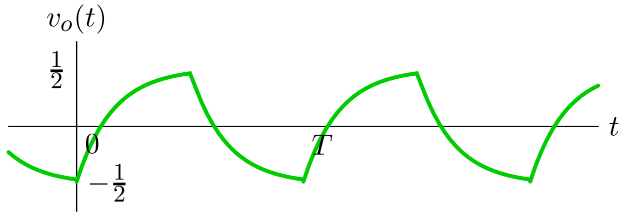
The point of this question is to understand how the magnitude of a filter affects the output and how the angle of a filter affects the output. Consider the following RC circuit as a "filter."



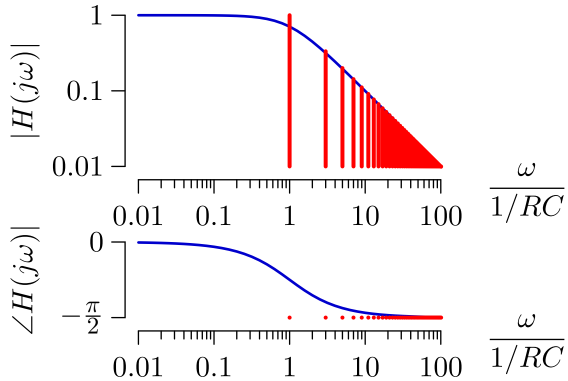
Assume that the input $v_i(t)$ is the following square wave.



If the fundamental frequency of the square wave ($\frac{2\pi}{T}$) is equal to the cutoff frequency of the RC circuit ($\frac{1}{RC}$) then the output $v_o(t)$ will have the following form.



We can think of the RC circuit as "filtering" the square wave as shown below.



The RC filter has two effects: (1) The amplitudes of the Fourier components of the input (vertical red lines in upper panel) are multiplied by the magnitude of the frequency response ($|H(j\omega)|$). (2) The phase of the Fourier components (red dots in lower panel) are shifted by the phase of the frequency response ($\angle H(j\omega)$).

a. Determine (using whatever method you find convenient) the output that would result if $v_i(t)$ were passed through a filter whose magnitude is $|H(j\omega)|$ (as above) but whose phase function is 0 for all frequencies. Compare the result with $v_o(t)$ above.

b. Determine (using whatever method you find convenient) the output that would result if $v_i(t)$ were passed through a filter whose phase function is $\angle H(j\omega)$ (as above) but whose magnitude function is 1 for all frequencies. Compare the result with $v_o(t)$ above.