6.003: Signals and Systems

Discrete-Time Systems

February 4, 2010

Discrete-Time Systems

We start with discrete-time (DT) systems because they

- are conceptually simpler than continuous-time systems
- illustrate same important modes of thinking as continuous-time
- are increasingly important (digital electronics and computation)

Multiple Representations of Discrete-Time Systems

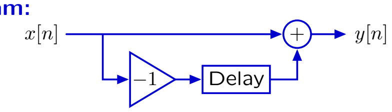
Systems can be represented in different ways to more easily address different types of issues.

Verbal description: 'To reduce the number of bits needed to store a sequence of large numbers that are nearly equal, record the first number, and then record successive differences.'

Difference equation:

$$y[n] = x[n] - x[n-1]$$

Block diagram:



We will exploit particular strengths of each of these representations.

Difference Equations

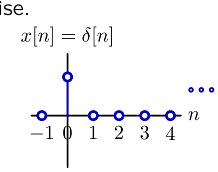
Difference equations are mathematically precise and compact.

Example:

$$y[n] = x[n] - x[n-1]$$

Let $x[n]$ equal the "unit sample" signal $\delta[n]$,

$$\delta[n] = \begin{cases} 1, & \text{if } n = 0; \\ 0, & \text{otherwise.} \end{cases}$$



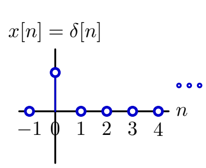
We will use the unit sample as a "primitive" (building-block signal) to construct more complex signals.

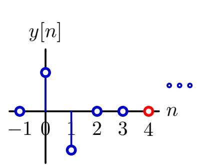
Step-By-Step Solutions

Difference equations are convenient for step-by-step analysis.

Find $y[n]$ given $x[n] = \delta[n]$: $y[n] = x[n] - x[n-1]$

$$y[-1] = x[-1] - x[-2] = 0 - 0 = 0$$
$$y[0] = x[0] - x[-1] = 1 - 0 = 1$$
$$y[1] = x[1] - x[0] = 0 - 1 = -1$$
$$y[2] = x[2] - x[1] = 0 - 0 = 0$$
$$y[3] = x[3] - x[2] = 0 - 0 = 0$$
$$\ldots$$

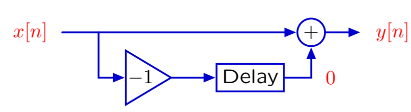


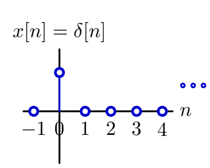


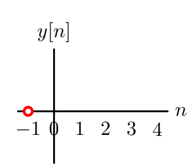
Step-By-Step Solutions

Block diagrams are also useful for step-by-step analysis.

Represent $y[n] = x[n] - x[n-1]$ with a block diagram: start "at rest"







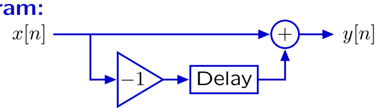
Check Yourself

DT systems can be described by difference equations and/or block diagrams.

Difference equation:

$y[n] = x[n] - x[n-1]$

Block diagram:



In what ways are these representations different?

From Samples to Signals

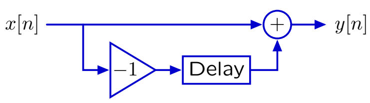
Lumping all of the (possibly infinite) samples into a single object — the signal — simplifies its manipulation.

This lumping is an **abstraction** that is analogous to

- representing coordinates in three-space as points
- representing lists of numbers as vectors in linear algebra
- creating an object in Python

From Samples to Signals

Operators manipulate signals rather than individual samples.



Nodes represent whole signals (e.g., X and Y).
The boxes **operate** on those signals:

- Delay = shift whole signal to right 1 time step
- Add = sum two signals
- -1: multiply by -1

Signals are the primitives.
Operators are the means of combination.

Operator Notation

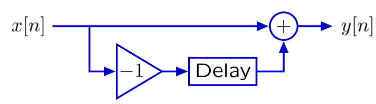
Symbols can now compactly represent diagrams.

Let $\mathcal{R}$ represent the right-shift **operator**:

$Y = \mathcal{R}\{X\} \equiv \mathcal{R}X$

where X represents the whole input signal ($x[n]$ for all n) and Y represents the whole output signal ($y[n]$ for all n)

Representing the difference machine



with $\mathcal{R}$ leads to the equivalent representation

$Y = X - \mathcal{R}X = (1 - \mathcal{R})\,X$

Operator Notation: Check Yourself

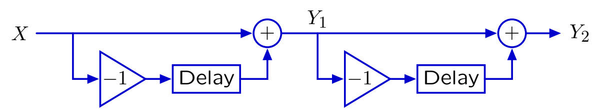
Let $Y = \mathcal{R}X$. Which of the following is/are true:

1. $y[n] = x[n]$ for all n
2. $y[n+1] = x[n]$ for all n
3. $y[n] = x[n+1]$ for all n
4. $y[n-1] = x[n]$ for all n
5. none of the above

Operator Representation of a Cascaded System

System operations have simple operator representations.

Cascade systems → multiply operator expressions.



Using operator notation:

$Y_1 = (1 - \mathcal{R})\,X$

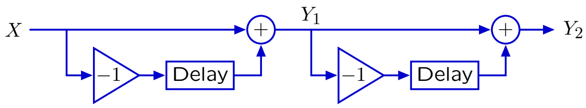
$Y_2 = (1 - \mathcal{R})\,Y_1$

Substituting for Y_1:

$Y_2 = (1 - \mathcal{R})(1 - \mathcal{R})\,X$

Operator Algebra

Operator expressions can be manipulated as polynomials.



Using difference equations:

$$y_2[n] = y_1[n] - y_1[n-1]$$
$$= (x[n] - x[n-1]) - (x[n-1] - x[n-2])$$
$$= x[n] - 2x[n-1] + x[n-2]$$

Using operator notation:

$$Y_2 = (1-\mathcal{R})\,Y_1 = (1-\mathcal{R})(1-\mathcal{R})\,X$$
$$= (1-\mathcal{R})^2 X$$
$$= (1-2\mathcal{R}+\mathcal{R}^2)\,X$$

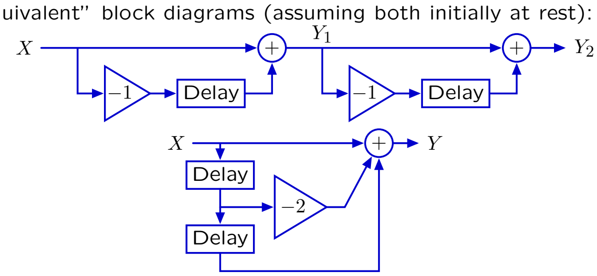
Operator Approach

Applies your existing expertise with polynomials to understand block diagrams, and thereby understand systems.

Operator Algebra

Operator notation facilitates seeing relations among systems.

"Equivalent" block diagrams (assuming both initially at rest):



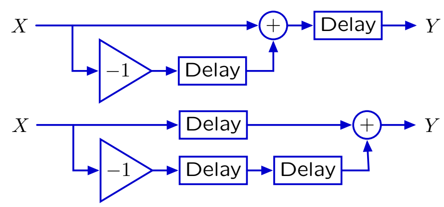
Equivalent operator expressions:

$$(1-\mathcal{R})(1-\mathcal{R}) = 1 - 2\mathcal{R} + \mathcal{R}^2$$

The operator equivalence is much easier to see.

Check Yourself

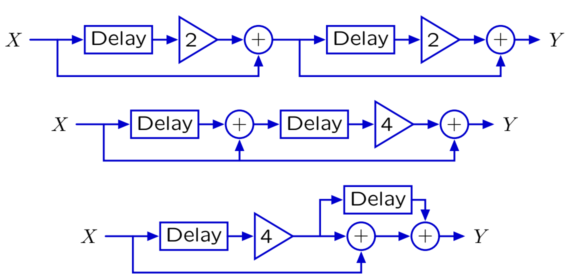
Operator expressions for these "equivalent" systems (if started "at rest") obey what mathematical property?



1. commutate
2. associative
3. distributive
4. transitive
5. none of the above

Check Yourself

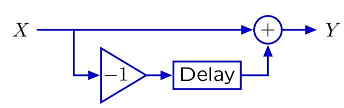
How many of the following systems are equivalent to $Y = (4\mathcal{R}^2 + 4\mathcal{R} + 1)\,X$?



Operator Algebra: Explicit and Implicit Rules

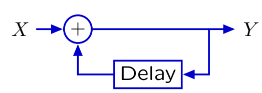
Recipes versus constraints.

Recipe: subtract a right-shifted version of the input signal from a copy of the input signal.



$$Y = (1-\mathcal{R})\,X$$

Constraint: the difference between Y and $\mathcal{R}Y$ is X.

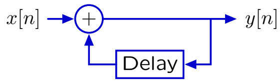


$$Y = \mathcal{R}Y + X$$
$$(1-\mathcal{R})\,Y = X$$

But how does one solve such a constraint?

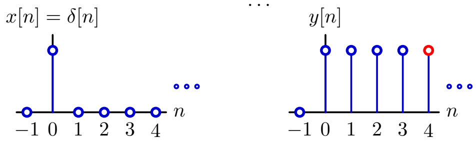
Example: Accumulator

Try step-by-step analysis: it always works. Start "at rest."



Find $y[n]$ given $x[n]=\delta[n]$: $y[n]=x[n]+y[n-1]$

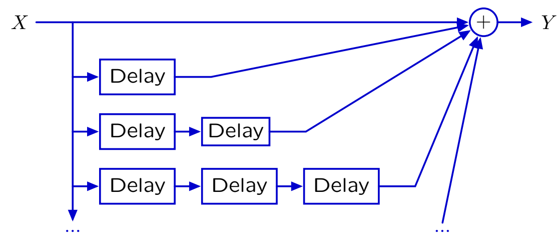
$$y[0]=x[0]+y[-1] \quad =1+0=1$$
$$y[1]=x[1]+y[0] \quad =0+1=1$$
$$y[2]=x[2]+y[1] \quad =0+1=1$$
$$\cdots$$



Persistent response to a transient input!

Example: Accumulator

The response of the accumulator system could also be generated by a system with infinitely many paths from input to output, each with one unit of delay more than the previous.



$$Y=(1+\mathcal{R}+\mathcal{R}^2+\mathcal{R}^3+\cdots)\,X$$

Example: Accumulator

These systems are equivalent in the sense that if each is initially at rest, they will produce identical outputs from the same input.

$$(1-\mathcal{R})\,Y_1=X_1 \quad \Leftrightarrow ? \quad Y_2=(1+\mathcal{R}+\mathcal{R}^2+\mathcal{R}^3+\cdots)\,X_2$$

Proof: Assume $X_2=X_1$:

$$\begin{aligned}
Y_2 &= (1+\mathcal{R}+\mathcal{R}^2+\mathcal{R}^3+\cdots)\,X_2\\
&= (1+\mathcal{R}+\mathcal{R}^2+\mathcal{R}^3+\cdots)\,X_1\\
&= (1+\mathcal{R}+\mathcal{R}^2+\mathcal{R}^3+\cdots)\,(1-\mathcal{R})\,Y_1\\
&= ((1+\mathcal{R}+\mathcal{R}^2+\mathcal{R}^3+\cdots)-(\mathcal{R}+\mathcal{R}^2+\mathcal{R}^3+\cdots))\,Y_1\\
&= Y_1
\end{aligned}$$

It follows that $Y_2=Y_1$.

It also follows that $(1-\mathcal{R})$ and $(1+\mathcal{R}+\mathcal{R}^2+\mathcal{R}^3+\cdots)$ are reciprocals.

Example: Accumulator

The reciprocal of $1-\mathcal{R}$ can also be evaluated using synthetic division.

$$\begin{array}{r|l}
 & 1 \;+\mathcal{R}\;+\mathcal{R}^2\;+\mathcal{R}^3\;+\cdots\\ \hline
1-\mathcal{R} & 1\\
 & 1\;-\mathcal{R}\\ \hline
 & \quad\mathcal{R}\\
 & \quad\mathcal{R}\;-\mathcal{R}^2\\ \hline
 & \qquad\mathcal{R}^2\\
 & \qquad\mathcal{R}^2\;-\mathcal{R}^3\\ \hline
 & \qquad\quad\mathcal{R}^3\\
 & \qquad\quad\mathcal{R}^3\;-\mathcal{R}^4\\ \hline
 & \qquad\qquad\cdots
\end{array}$$

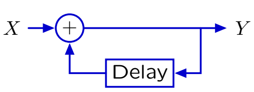
Therefore

$$\frac{1}{1-\mathcal{R}}=1+\mathcal{R}+\mathcal{R}^2+\mathcal{R}^3+\mathcal{R}^4+\cdots$$

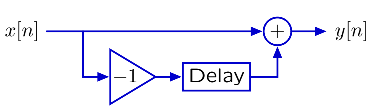
Feedback

Systems with signals that depend on previous values of the same signal are said to have **feedback**.

Example: The accumulator system has feedback.



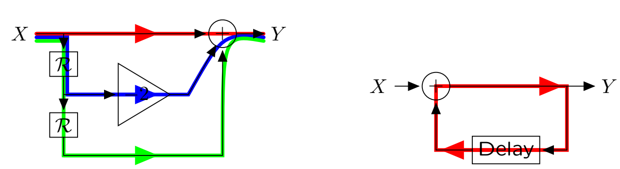
By contrast, the difference machine does not have feedback.



Cyclic Signal Paths, Feedback, and Modes

Block diagrams help visualize feedback.

Feedback occurs when there is a cyclic signal flow path.



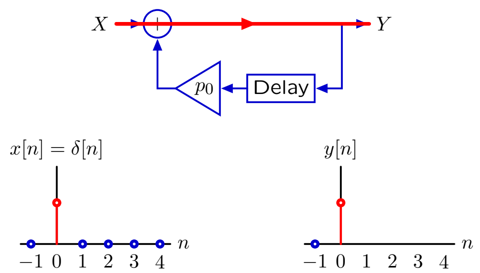
acyclic cyclic

Acyclic: all paths through system go from input to output with no cycles.

Cyclic: at least one cycle.

Feedback, Cyclic Signal Paths, and Modes

The effect of feedback can be visualized by tracing each cycle through the cyclic signal paths.



Each cycle creates another sample in the output.

Feedback, Cyclic Signal Paths, and Modes

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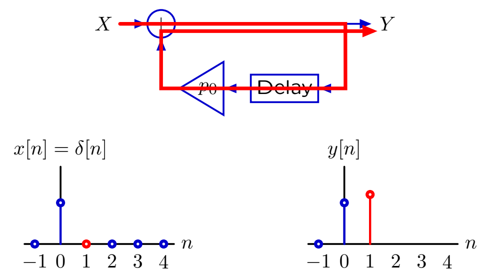

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Feedback, Cyclic Signal Paths, and Modes

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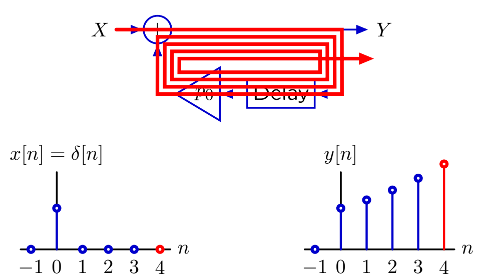
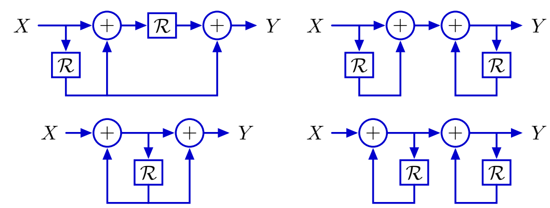


Each cycle creates another sample in the output.

The response will persist even though the input is transient.

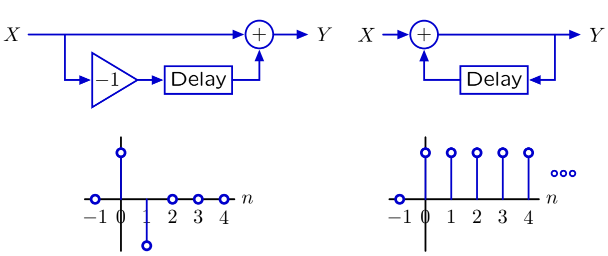
Check Yourself

How many of the following systems have cyclic signal paths?



Finite and Infinite Impulse Responses

The impulse response of an acyclic system has finite duration, while that of a cyclic system can have infinite duration.



Analysis of Cyclic Systems: Geometric Growth

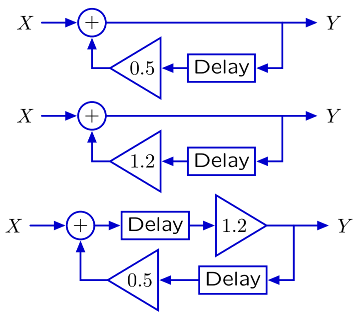
If traversing the cycle decreases or increases the magnitude of the signal, then the fundamental mode will decay or grow, respectively.

If the response decays toward zero, then we say that it **converges**.

Otherwise, we it **diverges**.

Check Yourself

How many of these systems have divergent unit-sample responses?



Cyclic Systems: Geometric Growth

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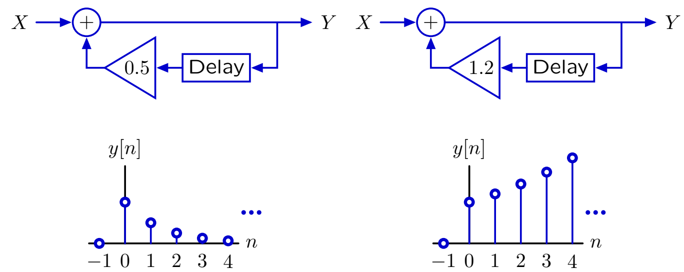


These are geometric sequences: $y[n] = (0.5)^n$ and $(1.2)^n$ for $n \geq 0$.

These geometric sequences are called **fundamental modes**.

Multiple Representations of Discrete-Time Systems

Now you know four representations of discrete-time systems.

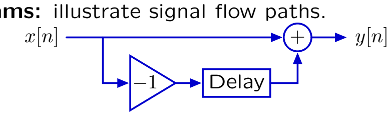
Verbal descriptions: preserve the rationale.

> "To reduce the number of bits needed to store a sequence of large numbers that are nearly equal, record the first number, and then record successive differences."

Difference equations: mathematically compact.

$$y[n] = x[n] - x[n-1]$$

Block diagrams: illustrate signal flow paths.



Operator representations: analyze systems as polynomials.

$$Y = (1 - \mathcal{R})\, X$$