6.003: Signals and Systems

Feedback, Poles, and Fundamental Modes

February 9, 2010

Last Time: Multiple Representations of DT Systems

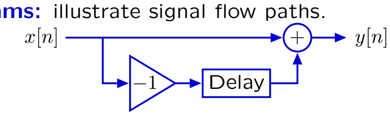
Verbal descriptions: preserve the rationale.

> "To reduce the number of bits needed to store a sequence of large numbers that are nearly equal, record the first number, and then record successive differences."

Difference equations: mathematically compact.

$$y[n] = x[n] - x[n-1]$$

Block diagrams: illustrate signal flow paths.



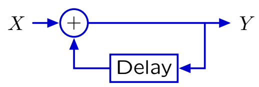
Operator representations: analyze systems as polynomials.

$$Y = (1 - \mathcal{R})\, X$$

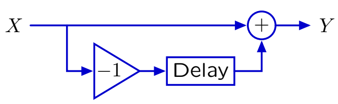
Last Time: Feedback, Cyclic Signal Paths, and Modes

Systems with signals that depend on previous values of the same signal are said to have **feedback**.

Example: The accumulator system has feedback.

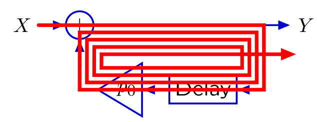


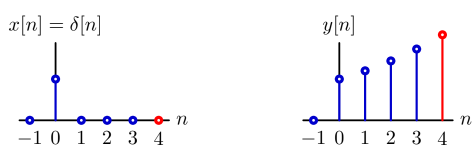
By contrast, the difference machine does not have feedback.



Last Time: Feedback, Cyclic Signal Paths, and Modes

The effect of feedback can be visualized by tracing each cycle through the cyclic signal paths.



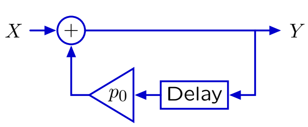


Each cycle creates another sample in the output.

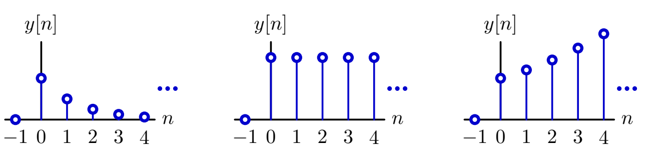
The response will persist even though the input is transient.

Geometric Growth: Poles

These unit-sample responses can be characterized by a single number — the **pole** — which is the base of the geometric sequence.



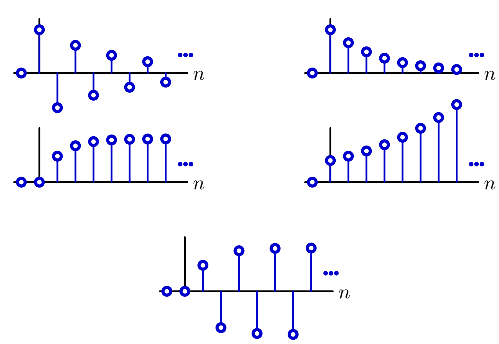
$$y[n] = \begin{cases} p_0^n, & \text{if } n >= 0; \\ 0, & \text{otherwise.} \end{cases}$$



$p_0 = 0.5$ $\quad$ $p_0 = 1$ $\quad$ $p_0 = 1.2$

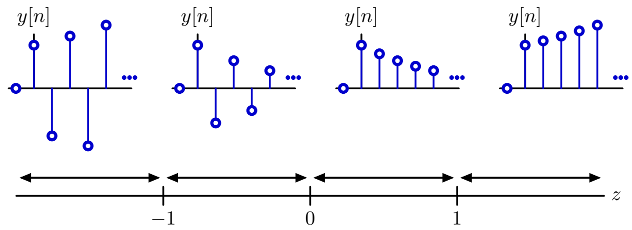
Check Yourself

How many of the following unit-sample responses can be represented by a single pole?



Geometric Growth

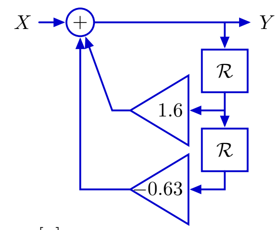
The value of p_0 determines the rate of growth.

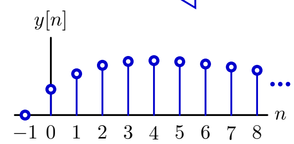


$p_0 < -1$: magnitude diverges, alternating sign
$-1 < p_0 < 0$: magnitude converges, alternating sign
$0 < p_0 < 1$: magnitude converges monotonically
$p_0 > 1$: magnitude diverges monotonically

Second-Order Systems

The unit-sample responses of more complicated cyclic systems are more complicated.

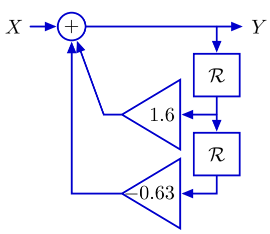




Not geometric. This response grows then decays.

Factoring Second-Order Systems

Factor the operator expression to break the system into two simpler systems (divide and conquer).



$$Y = X + 1.6\mathcal{R}Y - 0.63\mathcal{R}^2 Y$$

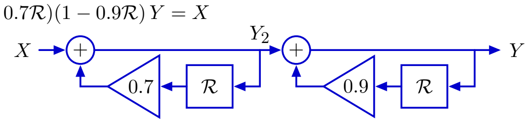
$$(1 - 1.6\mathcal{R} + 0.63\mathcal{R}^2)\,Y = X$$

$$(1 - 0.7\mathcal{R})(1 - 0.9\mathcal{R})\,Y = X$$

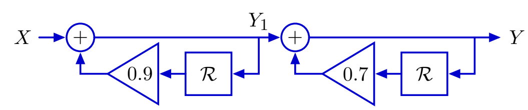
Factoring Second-Order Systems

The factored form corresponds to a cascade of simpler systems.

$$(1 - 0.7\mathcal{R})(1 - 0.9\mathcal{R})\,Y = X$$



$(1 - 0.7\mathcal{R})\,Y_2 = X$ $\quad$ $(1 - 0.9\mathcal{R})\,Y = Y_2$



$(1 - 0.9\mathcal{R})\,Y_1 = X$ $\quad$ $(1 - 0.7\mathcal{R})\,Y = Y_1$

The order doesn't matter (if systems are initially at rest).

Factoring Second-Order Systems

The unit-sample response of the cascaded system can be found by multiplying the polynomial representations of the subsystems.

$$\frac{Y}{X} = \frac{1}{(1 - 0.7\mathcal{R})(1 - 0.9\mathcal{R})} = \frac{1}{\underbrace{(1 - 0.7\mathcal{R})}} \times \frac{1}{\underbrace{(1 - 0.9\mathcal{R})}}$$

$$= \overbrace{(1 + 0.7\mathcal{R} + 0.7^2\mathcal{R}^2 + 0.7^3\mathcal{R}^3 + \cdots)} \times \overbrace{(1 + 0.9\mathcal{R} + 0.9^2\mathcal{R}^2 + 0.9^3\mathcal{R}^3 + \cdots)}$$

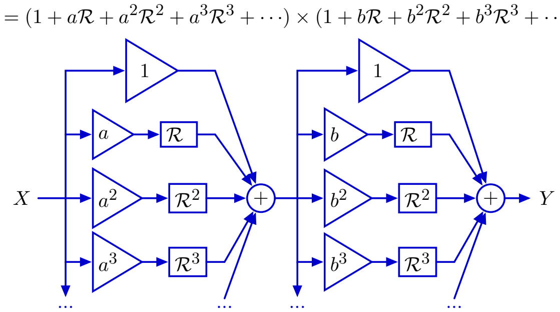
Multiply, then collect terms of equal order:

$$\frac{Y}{X} = 1 + (0.7 + 0.9)\mathcal{R} + (0.7^2 + 0.7 \times 0.9 + 0.9^2)\mathcal{R}^2$$
$$+ (0.7^3 + 0.7^2 \times 0.9 + 0.7 \times 0.9^2 + 0.9^3)\mathcal{R}^3 + \cdots$$

Multiplying Polynomial

Graphical representation of polynomial multiplication.

$$\frac{Y}{X} = (1 + a\mathcal{R} + a^2\mathcal{R}^2 + a^3\mathcal{R}^3 + \cdots) \times (1 + b\mathcal{R} + b^2\mathcal{R}^2 + b^3\mathcal{R}^3 + \cdots)$$



Collect terms of equal order:

$$\frac{Y}{X} = 1 + (a + b)\mathcal{R} + (a^2 + ab + b^2)\mathcal{R}^2 + (a^3 + a^2b + ab^2 + b^3)\mathcal{R}^3 + \cdots$$

Multiplying Polynomials

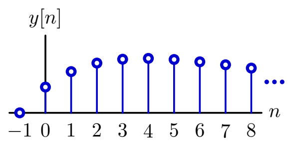
Tabular representation of polynomial multiplication.

$$(1+a\mathcal{R}+a^2\mathcal{R}^2+a^3\mathcal{R}^3+\cdots)\times(1+b\mathcal{R}+b^2\mathcal{R}^2+b^3\mathcal{R}^3+\cdots)$$

	1	$b\mathcal{R}$	$b^2\mathcal{R}^2$	$b^3\mathcal{R}^3$	$\cdots$
1	1	$b\mathcal{R}$	$b^2\mathcal{R}^2$	$b^3\mathcal{R}^3$	$\cdots$
$a\mathcal{R}$	$a\mathcal{R}$	$ab\mathcal{R}^2$	$ab^2\mathcal{R}^3$	$ab^3\mathcal{R}^4$	$\cdots$
$a^2\mathcal{R}^2$	$a^2\mathcal{R}^2$	$a^2b\mathcal{R}^3$	$a^2b^2\mathcal{R}^4$	$a^2b^3\mathcal{R}^5$	$\cdots$
$a^3\mathcal{R}^3$	$a^3\mathcal{R}^3$	$a^3b\mathcal{R}^4$	$a^3b^2\mathcal{R}^5$	$a^3b^3\mathcal{R}^6$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

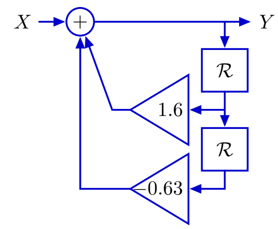
Group same powers of $\mathcal{R}$ by following reverse diagonals:

$$\frac{Y}{X}=1+(a+b)\mathcal{R}+(a^2+ab+b^2)\mathcal{R}^2+(a^3+a^2b+ab^2+b^3)\mathcal{R}^3+\cdots$$



Partial Fractions

Use partial fractions to rewrite as a **sum** of simpler parts.

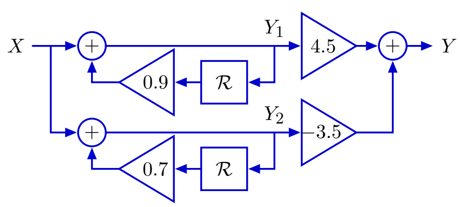


$$\frac{Y}{X}=\frac{1}{1-1.6\mathcal{R}+0.63\mathcal{R}^2}=\frac{1}{(1-0.9\mathcal{R})(1-0.7\mathcal{R})}=\frac{4.5}{1-0.9\mathcal{R}}-\frac{3.5}{1-0.7\mathcal{R}}$$

Second-Order Systems: Equivalent Forms

The sum of simpler parts suggests a parallel implementation.

$$\frac{Y}{X}=\frac{4.5}{1-0.9\mathcal{R}}-\frac{3.5}{1-0.7\mathcal{R}}$$



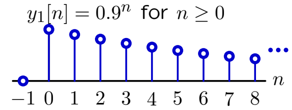
If $x[n]=\delta[n]$ then $y_1[n]=0.9^n$ and $y_2[n]=0.7^n$ for $n\geq 0$.

Thus, $y[n]=4.5(0.9)^n-3.5(0.7)^n$ for $n\geq 0$.

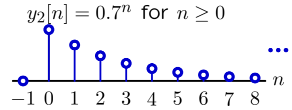
Partial Fractions

Graphical representation of the **sum** of geometric sequences.

$y_1[n]=0.9^n$ for $n\geq 0$

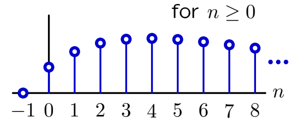


$y_2[n]=0.7^n$ for $n\geq 0$



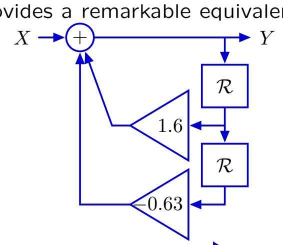
$y[n]=4.5(0.9)^n-3.5(0.7)^n$

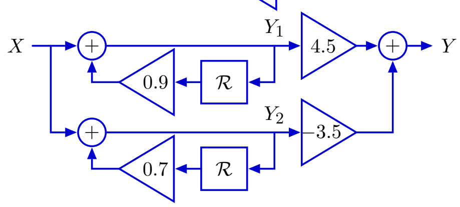
for $n\geq 0$



Partial Fractions

Partial fractions provides a remarkable equivalence.





→ follows from thinking about system as polynomial (factoring).

Poles

The key to simplifying a higher-order system is identifying its **poles**.

Poles are the roots of the denominator of the system functional when $\mathcal{R}\to\frac{1}{z}$.

Start with system functional:

$$\frac{Y}{X}=\frac{1}{1-1.6\mathcal{R}+0.63\mathcal{R}^2}=\frac{1}{(1-p_0\mathcal{R})(1-p_1\mathcal{R})}=\frac{1}{\underbrace{(1-0.7\mathcal{R})}_{p_0=0.7}\underbrace{(1-0.9\mathcal{R})}_{p_1=0.9}}$$

Substitute $\mathcal{R}\to\frac{1}{z}$ and find roots of denominator:

$$\frac{Y}{X}=\frac{1}{1-\frac{1.6}{z}+\frac{0.63}{z^2}}=\frac{z^2}{z^2-1.6z+0.63}=\frac{z^2}{\underbrace{(z-0.7)}_{z_0=0.7}\underbrace{(z-0.9)}_{z_1=0.9}}$$

The poles are at 0.7 and 0.9.

Check Yourself

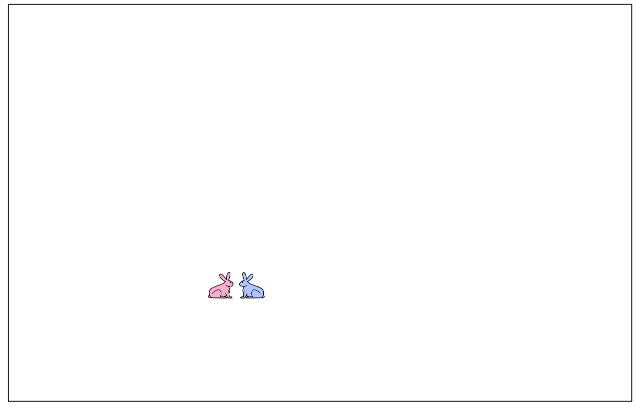
Consider the system described by

$$y[n] = -\frac{1}{4}y[n-1] + \frac{1}{8}y[n-2] + x[n-1] - \frac{1}{2}x[n-2]$$

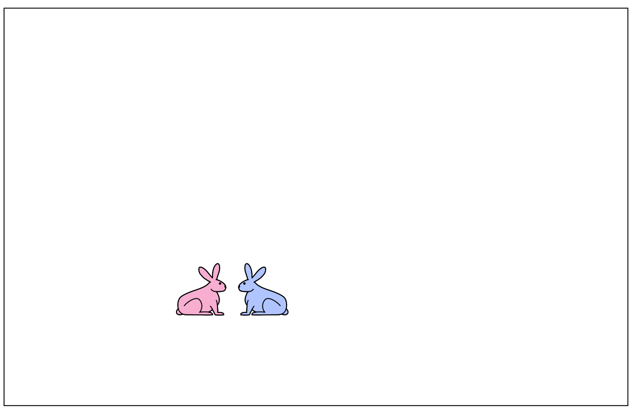
How many of the following are true?

1. The unit sample response converges to zero.
2. There are poles at $z = \frac{1}{2}$ and $z = \frac{1}{4}$.
3. There is a pole at $z = \frac{1}{2}$.
4. There are two poles.
5. None of the above

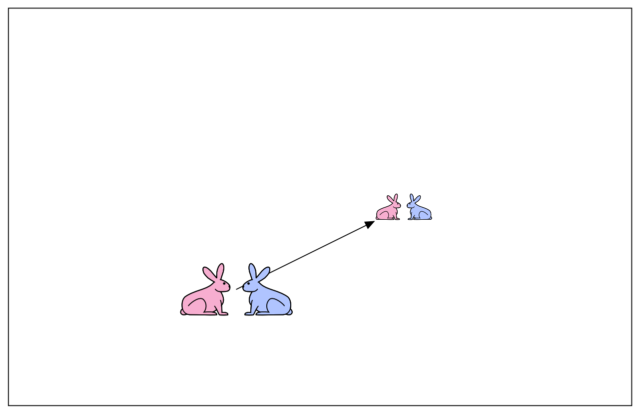
Population Growth



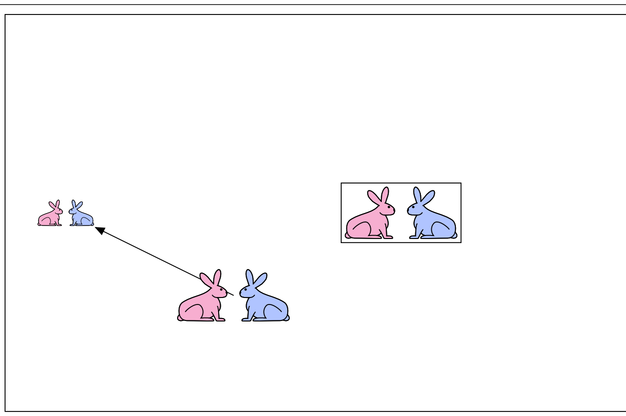
Population Growth



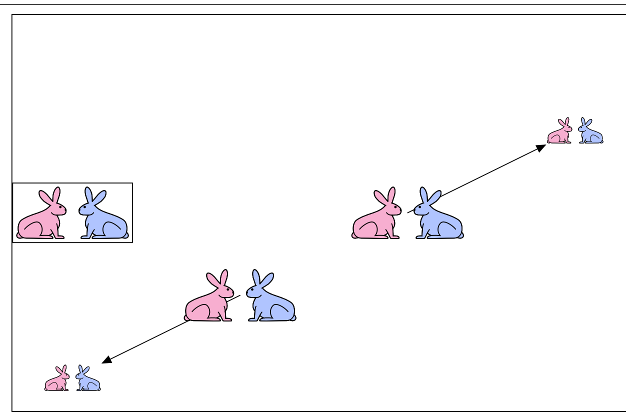
Population Growth



Population Growth



Population Growth



Check Yourself

What are the pole(s) of the Fibonacci system?

1. 1
2. 1 and -1
3. -1 and -2
4. $1.618\ldots$ and $-0.618\ldots$
5. none of the above

Example: Fibonacci's Bunnies

The unit-sample response of the Fibonacci system can be written as a weighted sum of fundamental modes.

$$H = \frac{Y}{X} = \frac{1}{1-\mathcal{R}-\mathcal{R}^2} = \frac{\frac{\phi}{\sqrt{5}}}{1-\phi\mathcal{R}} + \frac{\frac{1}{\phi\sqrt{5}}}{1+\frac{1}{\phi}\mathcal{R}}$$

$$h[n] = \frac{\phi}{\sqrt{5}}\phi^n + \frac{1}{\phi\sqrt{5}}(-\phi)^{-n}; \quad n \ge 0$$

But we already know that $h[n]$ is the Fibonacci sequence f:

$$f: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$$

Therefore we can calculate $f[n]$ without knowing $f[n-1]$ or $f[n-2]$!

Complex Poles

What if a pole has a non-zero imaginary part?

Example:

$$\frac{Y}{X} = \frac{1}{1-\mathcal{R}+\mathcal{R}^2}$$
$$= \frac{1}{1-\frac{1}{z}+\frac{1}{z^2}} = \frac{z^2}{z^2-z+1}$$

Poles are $z = \frac{1}{2} \pm \frac{\sqrt{3}}{2}j = e^{\pm j\pi/3}$.

What are the implications of complex poles?

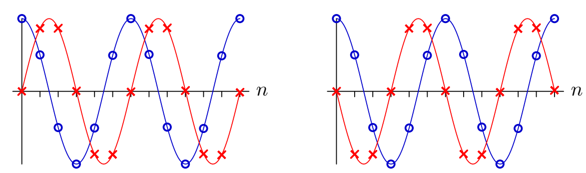
Complex Poles

Partial fractions work even when the poles are complex.

$$\frac{Y}{X} = \frac{1}{1-e^{j\pi/3}\mathcal{R}} \times \frac{1}{1-e^{-j\pi/3}\mathcal{R}} = \frac{1}{j\sqrt{3}}\left(\frac{e^{j\pi/3}}{1-e^{j\pi/3}\mathcal{R}} - \frac{e^{-j\pi/3}}{1-e^{-j\pi/3}\mathcal{R}}\right)$$

There are two fundamental modes (both geometric sequences):

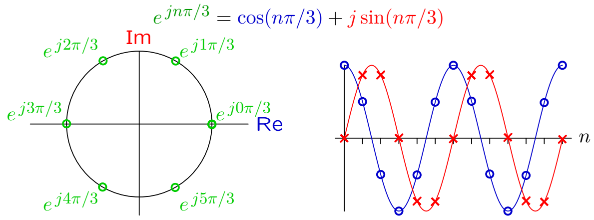
$e^{jn\pi/3} = \cos(n\pi/3) + j\sin(n\pi/3)$ and $e^{-jn\pi/3} = \cos(n\pi/3) - j\sin(n\pi/3)$



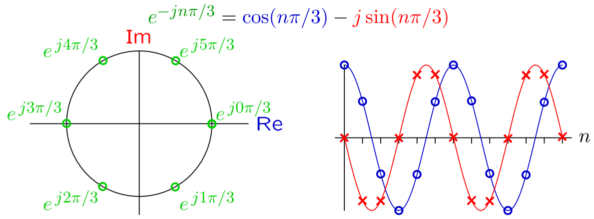
Complex Poles

Complex modes are easier to visualize in the complex plane.

$e^{jn\pi/3} = \cos(n\pi/3) + j\sin(n\pi/3)$



$e^{-jn\pi/3} = \cos(n\pi/3) - j\sin(n\pi/3)$



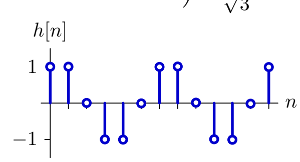
Complex Poles

The output of a "real" system has real values.

$$y[n] = x[n] + y[n-1] - y[n-2]$$

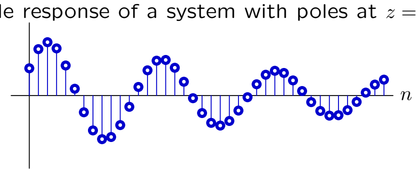
$$H = \frac{Y}{X} = \frac{1}{1-\mathcal{R}+\mathcal{R}^2}$$
$$= \frac{1}{1-e^{j\pi/3}\mathcal{R}} \times \frac{1}{1-e^{-j\pi/3}\mathcal{R}}$$
$$= \frac{1}{j\sqrt{3}}\left(\frac{e^{j\pi/3}}{1-e^{j\pi/3}\mathcal{R}} - \frac{e^{-j\pi/3}}{1-e^{-j\pi/3}\mathcal{R}}\right)$$

$$h[n] = \frac{1}{j\sqrt{3}}\left(e^{j(n+1)\pi/3} - e^{-j(n+1)\pi/3}\right) = \frac{2}{\sqrt{3}}\sin\frac{(n+1)\pi}{3}$$



Check Yourself

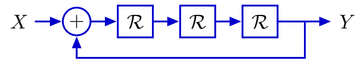
Unit-sample response of a system with poles at $z = re^{\pm j\Omega}$.



Which of the following is/are true?

1. $r < 0.5$ and $\Omega \approx 0.5$
2. $0.5 < r < 1$ and $\Omega \approx 0.5$
3. $r < 0.5$ and $\Omega \approx 0.08$
4. $0.5 < r < 1$ and $\Omega \approx 0.08$
5. none of the above

Check Yourself



How many of the following statements are true?

1. This system has 3 fundamental modes.
2. All of the fundamental modes can be written as geometrics.
3. Unit-sample response is $y[n] : 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1 \ldots$
4. Unit-sample response is $y[n] : 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1 \ldots$
5. One of the fundamental modes of this system is the unit step.

Summary

Systems composed of adders, gains, and delays can be characterized by their poles.

The poles of a system determine its fundamental modes.

The unit-sample response of a system can be expressed as a weighted sum of fundamental modes.

These properties follow from a polynomial interpretation of the system functional.