6.003: Signals and Systems

Continuous-Time Systems

February 11, 2010

Previously: DT Systems

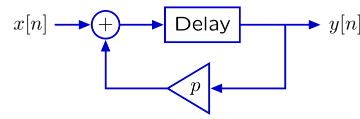
Verbal descriptions: preserve the rationale.

"Next year, your account will contain p times your balance from this year plus the money that you added this year."

Difference equations: mathematically compact.

$y[n+1] = x[n] + py[n]$

Block diagrams: illustrate signal flow paths.



Operator representations: analyze systems as polynomials.

$(1 - p\mathcal{R})\, Y = \mathcal{R}X$

Analyzing CT Systems

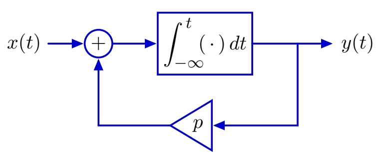
Verbal descriptions: preserve the rationale.

"Your account will grow in proportion to the current interest rate plus the rate at which you deposit."

Differential equations: mathematically compact.

$$\frac{dy(t)}{dt} = x(t) + py(t)$$

Block diagrams: illustrate signal flow paths.

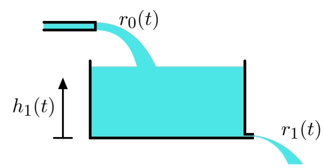


Operator representations: analyze systems as polynomials.

$(1 - p\mathcal{A})Y = \mathcal{A}X$

Differential Equations

Differential equations are mathematically precise and compact.



$$\frac{dr_1(t)}{dt} = \frac{r_0(t) - r_1(t)}{\tau}$$

Solution methodologies:

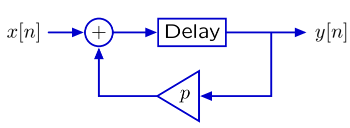
- general methods (separation of variables; integrating factors)
- homogeneous and particular solutions
- inspection

Today: new methods based on **block diagrams** and **operators**, which provide new ways to think about systems' behaviors.

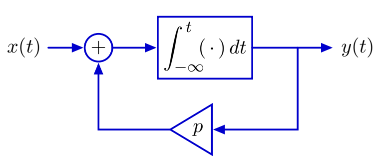
Block Diagrams

Block diagrams illustrate signal flow paths.

DT: adders, scalers, and delays – represent systems described by linear difference equations with constant coefficents.



CT: adders, scalers, and integrators – represent systems described by a linear differential equations with constant coefficients.



Operator Representation

CT Block diagrams are concisely represented with the $\mathcal{A}$ **operator**.

Applying $\mathcal{A}$ to a CT signal generates a new signal that is equal to the integral of the first signal at all points in time.

$Y = \mathcal{A}X$

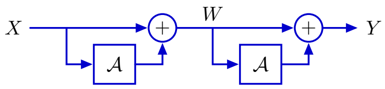
is equivalent to

$$y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$$

for **all** time t.

Evaluating Operator Expressions

As with $\mathcal{R}$, $\mathcal{A}$ expressions can be manipulated as polynomials.



$$w(t) = x(t) + \int_{-\infty}^{t} x(\tau)d\tau$$
$$y(t) = w(t) + \int_{-\infty}^{t} w(\tau)d\tau$$
$$y(t) = x(t) + \int_{-\infty}^{t} x(\tau)d\tau + \int_{-\infty}^{t} x(\tau)d\tau + \int_{-\infty}^{t} \left(\int_{-\infty}^{\tau_2} x(\tau_1)d\tau_1 \right) d\tau_2$$

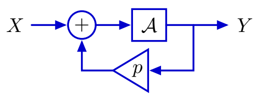
$$W = (1 + \mathcal{A})\, X$$
$$Y = (1 + \mathcal{A})\, W = (1 + \mathcal{A})(1 + \mathcal{A})\, X = (1 + 2\mathcal{A} + \mathcal{A}^2)\, X$$

Evaluating Operator Expressions

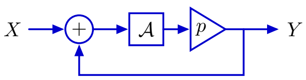
Expressions in $\mathcal{A}$ can be manipulated using rules for polynomials.

- Commutativity: $\mathcal{A}(1 - \mathcal{A})X = (1 - \mathcal{A})\mathcal{A}X$
- Distributivity: $\mathcal{A}(1 - \mathcal{A})X = (\mathcal{A} - \mathcal{A}^2)X$
- Associativity: $\Big((1 - \mathcal{A})\mathcal{A}\Big)(2 - \mathcal{A})X = (1 - \mathcal{A})\Big(\mathcal{A}(2 - \mathcal{A})\Big)X$

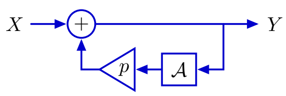
Check Yourself



$\dot{y}(t) = \dot{x}(t) + p\ddot{y}(t)$



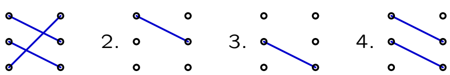
$\dot{y}(t) = x(t) + py(t)$



$\dot{y}(t) = px(t) + py(t)$

Which best illustrates the left-right correspondences?

1. 2. 3. 4.

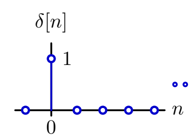


5. none

Elementary Building-Block Signals

Elementary DT signal: $\delta[n]$.

$$\delta[n] = \begin{cases} 1, & \text{if } n = 0; \\ 0, & \text{otherwise} \end{cases}$$



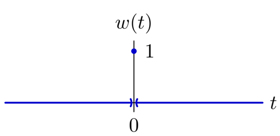
- shortest possible duration (most "transient")
- useful for constructing more complex signals

What CT signal serves the same purpose?

Elementary CT Building-Block Signal

Consider the analogous CT signal.

$$w(t) = \begin{cases} 0 & t < 0 \\ 1 & t = 0 \\ 0 & t > 0 \end{cases}$$

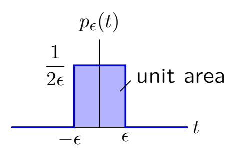


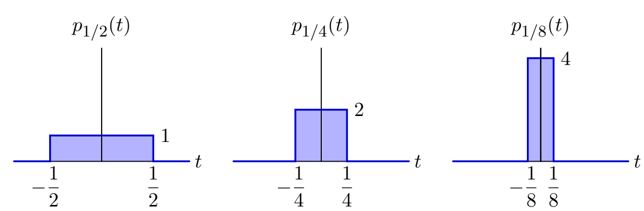
Is this a good choice as a building-block signal?

Unit-Impulse Signal

The unit-impulse signal acts as a pulse with unit area but zero width.

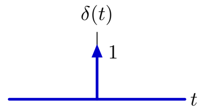
$$\delta(t) = \lim_{\epsilon \to 0} p_\epsilon(t)$$





Unit-Impulse Signal

The unit-impulse function is represented by an arrow with the number **1**, which represents its area or "weight."

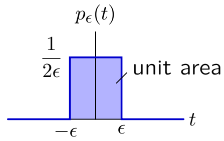


It has two seemingly contradictory properties:

- it is nonzero only at $t = 0$, and
- its definite integral $(-\infty, \infty)$ is one!

Both of these properties follow from thinking about $\delta(t)$ as a limit:

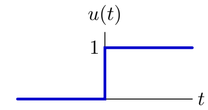
$$\delta(t) = \lim_{\epsilon \to 0} p_\epsilon(t)$$



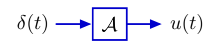
Unit-Impulse and Unit-Step Signals

The indefinite integral of the unit-impulse is the unit-step.

$$u(t) = \int_{-\infty}^{t} \delta(\lambda)\, d\lambda = \begin{cases} 1; & t \geq 0 \\ 0; & \text{otherwise} \end{cases}$$

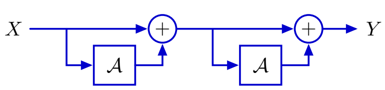


Equivalently



Impulse Response of Acyclic CT System

If the block diagram of a CT system has no feedback (i.e., no cycles), then the corresponding operator expression is "imperative."



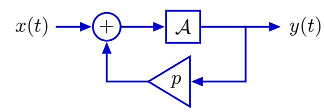
$$Y = (1 + \mathcal{A})(1 + \mathcal{A})\, X = (1 + 2\mathcal{A} + \mathcal{A}^2)\, X$$

If $x(t) = \delta(t)$ then

$$y(t) = (1 + 2\mathcal{A} + \mathcal{A}^2)\, \delta(t) = \delta(t) + 2u(t) + tu(t)$$

CT Feedback

Find the impulse response of this CT system with feedback.



Method 1: find differential equation and solve it.

$$\dot{y}(t) = x(t) + py(t)$$

Linear, first-order difference equation with constant coefficients.

Try $y(t) = Ce^{\alpha t}u(t)$.

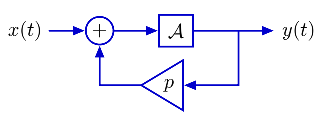
Then $\dot{y}(t) = \alpha Ce^{\alpha t}u(t) + Ce^{\alpha t}\delta(t) = \alpha Ce^{\alpha t}u(t) + C\delta(t)$.

Substituting, we find that $\alpha Ce^{\alpha t}u(t) + C\delta(t) = \delta(t) + pCe^{\alpha t}u(t)$.

Therefore $\alpha = p$ and $C = 1$ $\rightarrow$ $y(t) = e^{pt}u(t)$.

CT Feedback

Find the impulse response of this CT system with feedback.



Method 2: use operators.

$$Y = \mathcal{A}\,(X + pY)$$
$$\frac{Y}{X} = \frac{\mathcal{A}}{1 - p\mathcal{A}}$$

Now expand in ascending series in $\mathcal{A}$:

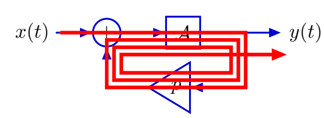
$$\frac{Y}{X} = \mathcal{A}(1 + p\mathcal{A} + p^2\mathcal{A}^2 + p^3\mathcal{A}^3 + \cdots)$$

If $x(t) = \delta(t)$ then

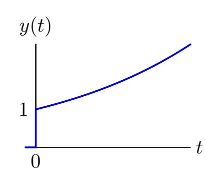
$$y(t) = \mathcal{A}(1 + p\mathcal{A} + p^2\mathcal{A}^2 + p^3\mathcal{A}^3 + \cdots)\, \delta(t)$$
$$= (1 + pt + \frac{1}{2}p^2t^2 + \frac{1}{6}p^3t^3 + \cdots)\, u(t) = e^{pt}u(t)\,.$$

CT Feedback

We can visualize the feedback by tracing each cycle through the cyclic signal path.

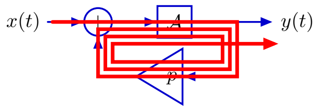


$$y(t) = (\mathcal{A} + p\mathcal{A}^2 + p^2\mathcal{A}^3 + p^3\mathcal{A}^4 + \cdots)\, \delta(t)$$
$$= (1 + pt + \frac{1}{2}p^2t^2 + \frac{1}{6}p^3t^3 + \cdots)\, u(t) = e^{pt}u(t)$$

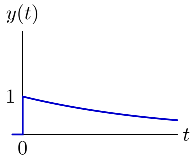


CT Feedback

Making p negative makes the output converge.

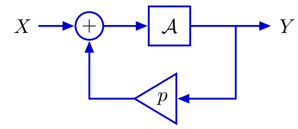


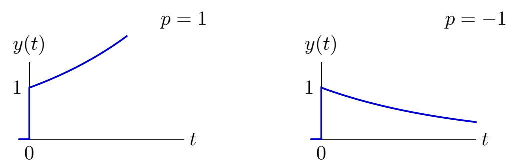
$$y(t) = (\mathcal{A} - p\mathcal{A}^2 + p^2\mathcal{A}^3 - p^3\mathcal{A}^4 + \cdots)\,\delta(t)$$
$$= (1 - pt + \frac{1}{2}p^2t^2 - \frac{1}{6}p^3t^3 + \cdots)\,u(t) = e^{-pt}u(t)$$



Convergent and Divergent Poles

The fundamental mode associated with p diverges if $p > 0$ and converges if $p < 0$.





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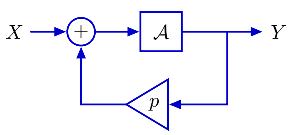
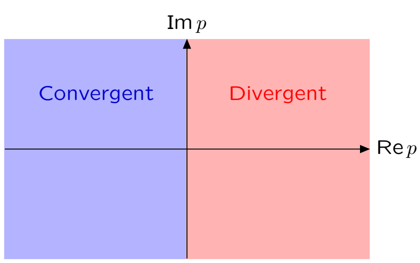
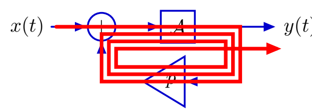


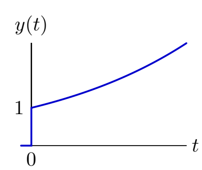


CT Feedback

In CT, each cycle adds a new integration.

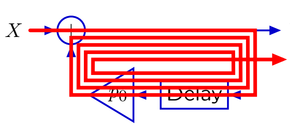


$$y(t) = (\mathcal{A} + p\mathcal{A}^2 + p^2\mathcal{A}^3 + p^3\mathcal{A}^4 + \cdots)\,\delta(t)$$
$$= (1 + pt + \frac{1}{2}p^2t^2 + \frac{1}{6}p^3t^3 + \cdots)\,u(t) = e^{pt}u(t)$$

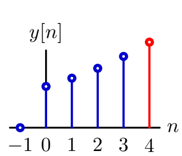


Feedback in DT Systems

In DT, each cycle creates another sample in the output.

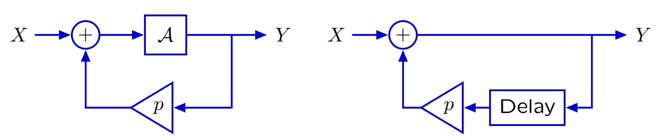


$$y[n] = (1 + p\mathcal{R} + p^2\mathcal{R}^2 + p^3\mathcal{R}^3 + p^4\mathcal{R}^4 + \cdots)\,\delta[n]$$
$$= \delta[n] + p\delta[n-1] + p^2\delta[n-2] + p^3\delta[n-3] + p^4\delta[n-4] + \cdots$$



Comparison of CT and DT representations

Locations of convergent poles differ for CT and DT systems.

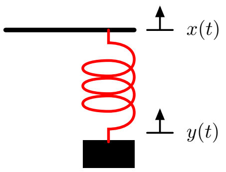


$$\frac{\mathcal{A}}{1 - p\mathcal{A}} \qquad \frac{1}{1 - p\mathcal{R}}$$

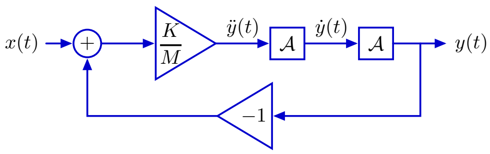
$$e^{pt}u(t) \qquad p^n u[n]$$

Mass and Spring System

Use the $\mathcal{A}$ operator to solve the mass and spring system.



$$F = K\big(x(t) - y(t)\big) = M\ddot{y}(t)$$



$$\frac{Y}{X} = \frac{\frac{K}{M}\mathcal{A}^2}{1 + \frac{K}{M}\mathcal{A}^2}$$

Mass and Spring System

Factor system functional to find the poles.

$$\frac{Y}{X} = \frac{\frac{K}{M}\mathcal{A}^2}{1 + \frac{K}{M}\mathcal{A}^2} = \frac{\frac{K}{M}\mathcal{A}^2}{(1 - p_0\mathcal{A})(1 - p_1\mathcal{A})}$$

$$1 + \frac{K}{M}\mathcal{A}^2 = 1 - (p_0 + p_1)\mathcal{A} + p_0 p_1 \mathcal{A}^2$$

The sum of the poles must be zero.
The product of the poles must be K/M.

$$p_0 = j\sqrt{\frac{K}{M}} \quad p_1 = -j\sqrt{\frac{K}{M}}$$

Mass and Spring System

Alternatively, find the poles by substituting $\mathcal{A} \rightarrow \frac{1}{s}$.
The poles are then the roots of the denominator.

$$\frac{Y}{X} = \frac{\frac{K}{M}\mathcal{A}^2}{1 + \frac{K}{M}\mathcal{A}^2}$$

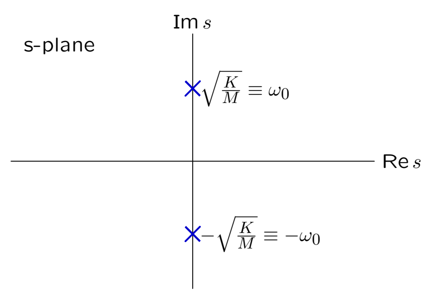
Substitute $\mathcal{A} \rightarrow \frac{1}{s}$:

$$\frac{Y}{X} = \frac{\frac{K}{M}}{s^2 + \frac{K}{M}}$$

$$s = \pm j\sqrt{\frac{K}{M}}$$

Mass and Spring System

The poles are complex conjugates.



The corresponding fundamental modes have complex values.

fundamental mode 1: $e^{j\omega_0 t} = \cos\omega_0 t + j\sin\omega_0 t$

fundamental mode 2: $e^{-j\omega_0 t} = \cos\omega_0 t - j\sin\omega_0 t$

Mass and Spring System

Real-valued inputs always excite combinations of these modes so that the imaginary parts cancel.

Example: find the impulse response.

$$\frac{Y}{X} = \frac{\frac{K}{M}\mathcal{A}^2}{1 + \frac{K}{M}\mathcal{A}^2} = \frac{\frac{K}{M}}{p_0 - p_1}\left(\frac{\mathcal{A}}{1 - p_0\mathcal{A}} - \frac{\mathcal{A}}{1 - p_1\mathcal{A}}\right)$$
$$= \frac{\omega_0^2}{2j\omega_0}\left(\frac{\mathcal{A}}{1 - j\omega_0\mathcal{A}} - \frac{\mathcal{A}}{1 + j\omega_0\mathcal{A}}\right)$$
$$= \underbrace{\frac{\omega_0}{2j}\left(\frac{\mathcal{A}}{1 - j\omega_0\mathcal{A}}\right)}_{\text{makes mode 1}} - \underbrace{\frac{\omega_0}{2j}\left(\frac{\mathcal{A}}{1 + j\omega_0\mathcal{A}}\right)}_{\text{makes mode 2}}$$

The modes themselves are complex conjugates, and their coefficients are also complex conjugates. So the sum is a sum of something and its complex conjugate, which is real.

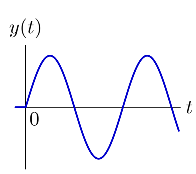
Mass and Spring System

The impulse response is therefore real.

$$\frac{Y}{X} = \frac{\omega_0}{2j}\left(\frac{\mathcal{A}}{1 - j\omega_0\mathcal{A}}\right) - \frac{\omega_0}{2j}\left(\frac{\mathcal{A}}{1 + j\omega_0\mathcal{A}}\right)$$

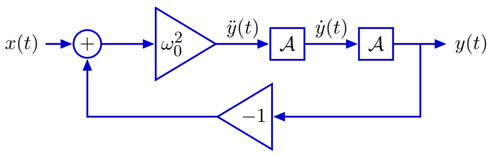
The impulse response is

$$h(t) = \frac{\omega_0}{2j}e^{j\omega_0 t} - \frac{\omega_0}{2j}e^{-j\omega_0 t} = \omega_0 \sin\omega_0 t\,; \qquad t > 0$$



Mass and Spring System

Alternatively, find impulse response by expanding system functional.



$$\frac{Y}{X} = \frac{\omega_0^2\mathcal{A}^2}{1+\omega_0^2\mathcal{A}^2} = \omega_0^2\mathcal{A}^2 - \omega_0^4\mathcal{A}^4 + \omega_0^6\mathcal{A}^6 - + \cdots$$

If $x(t) = \delta(t)$ then

$$y(t) = \omega_0^2 t - \omega_0^4 \frac{t^3}{3!} + \omega_0^6 \frac{t^5}{5!} - + \cdots \;,\; t \ge 0$$

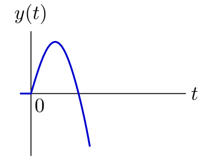
Mass and Spring System

Look at successive approximations to this infinite series.

$$\frac{Y}{X} = \frac{\omega_0^2\mathcal{A}^2}{1+\omega_0^2\mathcal{A}^2} = \omega_0^2\mathcal{A}^2 \sum_{l=0}^{\infty} \left(-\omega_0^2\mathcal{A}^2\right)^l$$

If $x(t) = \delta(t)$ then

$$y(t) = \sum_{l=0}^{\infty} \omega_0^2 \left(-\omega_0^2\right)^l \mathcal{A}^{2l+2}\delta(t)$$
$$= \omega_0^2 t - \omega_0^4 \frac{t^3}{3!} + \omega_0^6 \frac{t^5}{5!} - \omega_0^8 \frac{t^7}{7!}$$



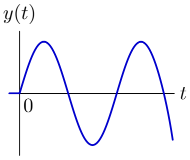
Mass and Spring System

Look at successive approximations to this infinite series.

$$\frac{Y}{X} = \frac{\omega_0^2\mathcal{A}^2}{1+\omega_0^2\mathcal{A}^2} = \omega_0^2\mathcal{A}^2 \sum_{l=0}^{\infty} \left(-\omega_0^2\mathcal{A}^2\right)^l$$

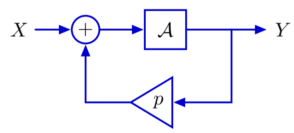
If $x(t) = \delta(t)$ then

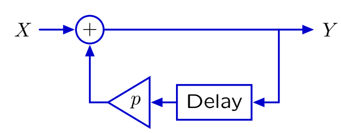
$$y(t) = \sum_{l=0}^{\infty} \omega_0^2 \left(-\omega_0^2\right)^l \mathcal{A}^{2l+2}\delta(t)$$
$$= \omega_0^2 t - \omega_0^4 \frac{t^3}{3!} + \omega_0^6 \frac{t^5}{5!} - \omega_0^8 \frac{t^7}{7!} + \omega_0^{10} \frac{t^9}{9!} - + \cdots = \omega_0 \sin \omega_0 t$$



Comparison of CT and DT representations

Important similarities and important differences.





$$\frac{\mathcal{A}}{1-p\mathcal{A}} \qquad \frac{1}{1-p\mathcal{R}}$$

$$e^{pt}u(t) \qquad p^n u[n]$$