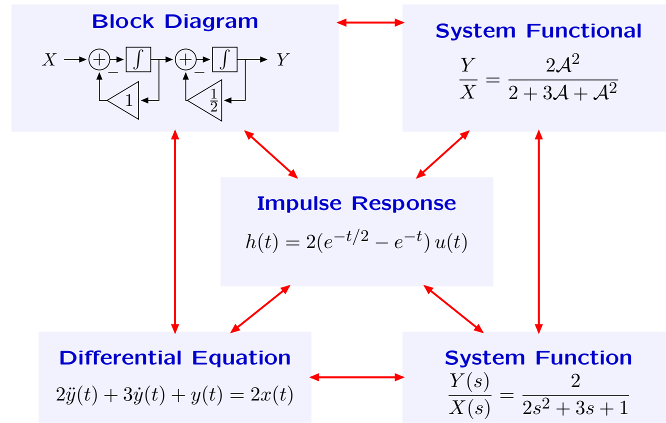
6.003: Signals and Systems

Laplace Transform

February 18, 2010

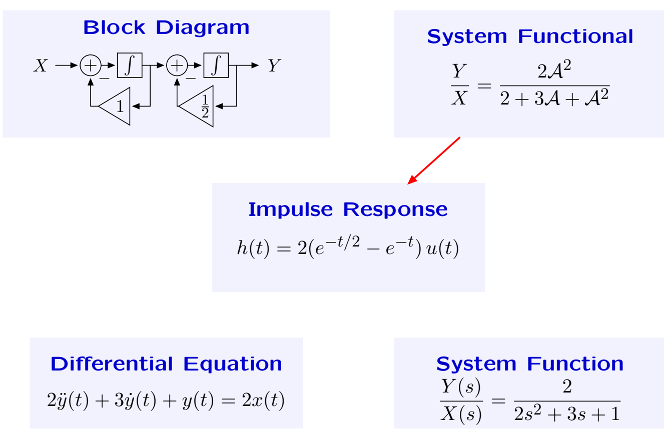
Concept Map: Continuous-Time Systems

Relations among representations.



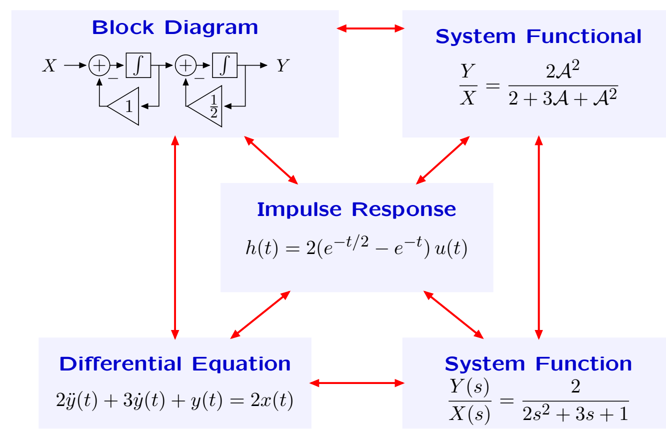
Check Yourself

How to determine impulse response from system functional?



Concept Map: Continuous-Time Systems

Today: new relations based on Laplace transform.



Laplace Transform: Definition

Laplace transform maps a function of time t to a function of s.

$$X(s) = \int x(t)e^{-st}dt$$

There are two important variants:

Unilateral (18.03)

$$X(s) = \int_0^\infty x(t)e^{-st}dt$$

Bilateral (6.003)

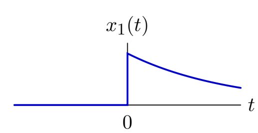
$$X(s) = \int_{-\infty}^\infty x(t)e^{-st}dt$$

Both share important properties — will discuss differences later.

Laplace Transforms

Example: Find the Laplace transform of $x_1(t)$:

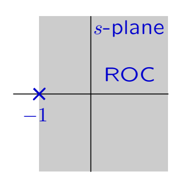
$$x_1(t) = \begin{cases} e^{-t} & \text{if } t \ge 0 \\ 0 & \text{otherwise} \end{cases}$$



$$X_1(s) = \int_{-\infty}^{\infty} x_1(t)e^{-st}dt = \int_0^\infty e^{-t}e^{-st}dt = \left.\frac{e^{-(s+1)t}}{-(s+1)}\right|_0^\infty = \frac{1}{s+1}$$

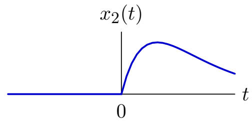
provided $\text{Re}(s+1) > 0$ which implies that $\text{Re}(s) > -1$.

$$\frac{1}{s+1}\ ;\quad \text{Re}(s) > -1$$



Check Yourself

$$x_2(t) = \begin{cases} e^{-t} - e^{-2t} & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$



Which of the following is the Laplace transform of $x_2(t)$?

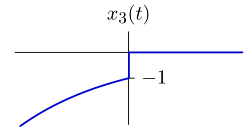
1. $X_2(s) = \frac{1}{(s+1)(s+2)}$; $\text{Re}(s) > -1$
2. $X_2(s) = \frac{1}{(s+1)(s+2)}$; $\text{Re}(s) > -2$
3. $X_2(s) = \frac{s}{(s+1)(s+2)}$; $\text{Re}(s) > -1$
4. $X_2(s) = \frac{s}{(s+1)(s+2)}$; $\text{Re}(s) > -2$
5. none of the above

Regions of Convergence

Left-sided signals have left-sided Laplace transforms (bilateral only).

Example:

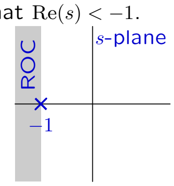
$$x_3(t) = \begin{cases} -e^{-t} & \text{if } t \leq 0 \\ 0 & \text{otherwise} \end{cases}$$



$$X_3(s) = \int_{-\infty}^{\infty} x_3(t) e^{-st} dt = \int_{-\infty}^{0} -e^{-t} e^{-st} dt = \left. \frac{-e^{-(s+1)t}}{-(s+1)} \right|_{-\infty}^{0} = \frac{1}{s+1}$$

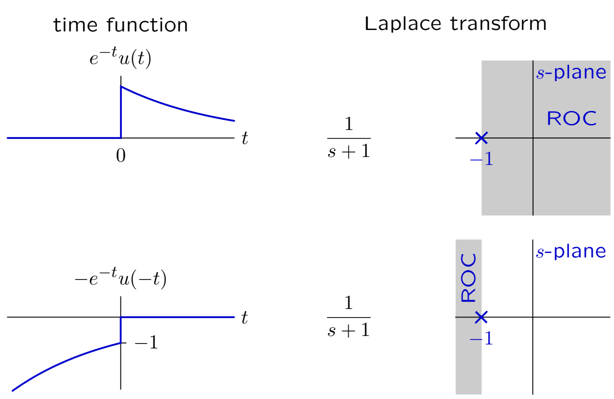
provided $\text{Re}(s+1) < 0$ which implies that $\text{Re}(s) < -1$.

$$\frac{1}{s+1} \; ; \quad \text{Re}(s) < -1$$



Left- and Right-Sided ROCs

Laplace transforms of left- and right-sided exponentials have the same form (except −); with left- and right-sided ROCs, respectively.



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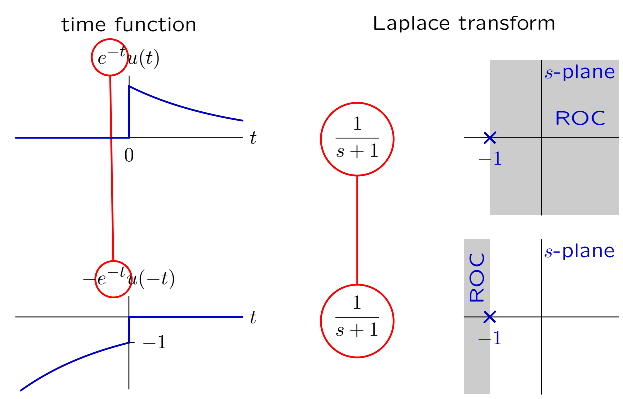
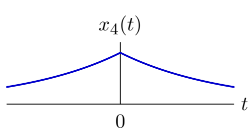


Check Yourself

Find the Laplace transform of $x_4(t)$.

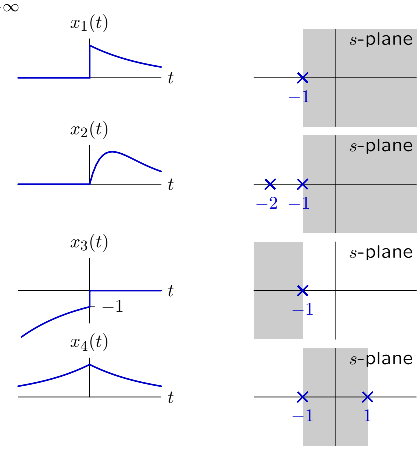
$$x_4(t) = e^{-|t|}$$



1. $X_4(s) = \frac{2}{1-s^2}$; $-\infty < \text{Re}(s) < \infty$
2. $X_4(s) = \frac{2}{1-s^2}$; $-1 < \text{Re}(s) < 1$
3. $X_4(s) = \frac{2}{1+s^2}$; $-\infty < \text{Re}(s) < \infty$
4. $X_4(s) = \frac{2}{1+s^2}$; $-1 < \text{Re}(s) < 1$
5. none of the above

Time-Domain Interpretation of ROC

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st} dt$$



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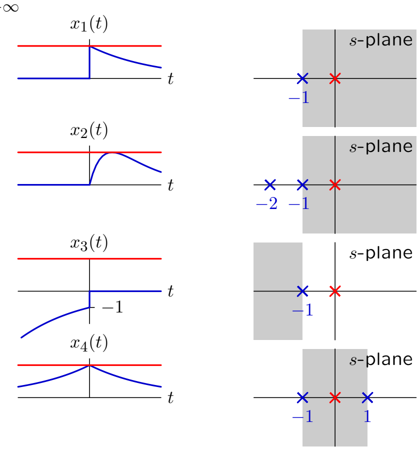

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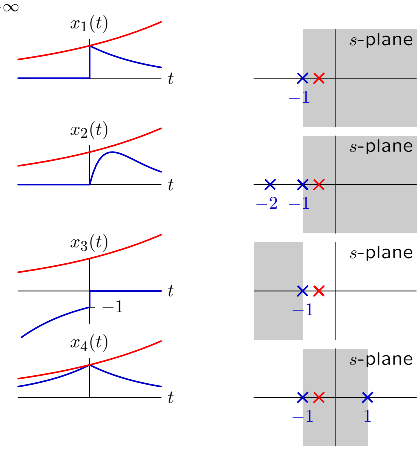

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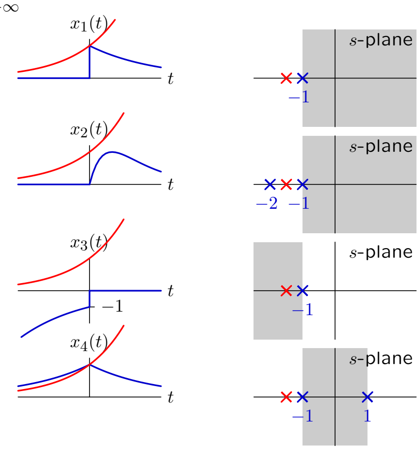


Check Yourself

The Laplace transform $\frac{2s}{s^2-4}$ corresponds to how many of the following signals?

1. $e^{-2t}u(t)+e^{2t}u(t)$
2. $e^{-2t}u(t)-e^{2t}u(-t)$
3. $-e^{-2t}u(-t)+e^{2t}u(t)$
4. $-e^{-2t}u(-t)-e^{2t}u(-t)$

Solving Differential Equations with Laplace Transforms

Solve the following differential equation:

$$\dot{y}(t)+y(t)=\delta(t)$$

Take the Laplace transform of this equation.

$$\mathcal{L}\{\dot{y}(t)+y(t)\}=\mathcal{L}\{\delta(t)\}$$

The Laplace transform of a sum is the sum of the Laplace transforms (prove this as an exercise).

$$\mathcal{L}\{\dot{y}(t)\}+\mathcal{L}\{y(t)\}=\mathcal{L}\{\delta(t)\}$$

What's the Laplace transform of a derivative?

Laplace transform of a derivative

Assume that $X(s)$ is the Laplace transform of $x(t)$:

$$X(s)=\int_{-\infty}^{\infty} x(t)e^{-st}dt$$

Find the Laplace transform of $y(t)=\dot{x}(t)$.

$$Y(s)=\int_{-\infty}^{\infty} y(t)e^{-st}dt=\int_{-\infty}^{\infty} \underbrace{\dot{x}(t)}_{\dot{v}}\,\underbrace{e^{-st}}_{u}\,dt$$

$$=\underbrace{x(t)}_{v}\,\underbrace{e^{-st}}_{u}\Big|_{-\infty}^{\infty}-\int_{-\infty}^{\infty}\underbrace{x(t)}_{v}\underbrace{(-se^{-st})}_{\dot{u}}dt$$

The first term must be zero since $X(s)$ converged. Thus

$$Y(s)=s\int_{-\infty}^{\infty} x(t)e^{-st}dt=sX(s)$$

Solving Differential Equations with Laplace Transforms

Back to the previous problem:

$$\mathcal{L}\{\dot{y}(t)\} + \mathcal{L}\{y(t)\} = \mathcal{L}\{\delta(t)\}$$

Let $Y(s)$ represent the Laplace transform of $y(t)$.

Then $sY(s)$ is the Laplace transform of $\dot{y}(t)$.

$$sY(s) + Y(s) = \mathcal{L}\{\delta(t)\}$$

What's the Laplace transform of the impulse function?

Laplace transform of the impulse function

Let $x(t) = \delta(t)$.

$$\begin{aligned} X(s) &= \int_{-\infty}^{\infty} \delta(t) e^{-st} dt \\ &= \int_{-\infty}^{\infty} \delta(t)\, e^{-st}\big|_{t=0}\, dt \\ &= \int_{-\infty}^{\infty} \delta(t)\ 1\ dt \\ &= 1 \end{aligned}$$

Sifting property: $\delta(t)$ **sifts** out the value of e^{-st} at $t = 0$.

Solving Differential Equations with Laplace Transforms

Back to the previous problem:

$$sY(s) + Y(s) = \mathcal{L}\{\delta(t)\} = 1$$

This is a simple algebraic expression. Solve for $Y(s)$:

$$Y(s) = \frac{1}{s+1}$$

We've seen this Laplace transform previously.

$$y(t) = e^{-t}u(t) \qquad \text{(why not } y(t) = -e^{-t}u(-t)\text{ ?)}$$

Notice that we solved the differential equation $\dot{y}(t)+y(t) = \delta(t)$ without computing homogeneous and particular solutions.

Solving Differential Equations with Laplace Transforms

Summary of method.

Start with differential equation:

$$\dot{y}(t) + y(t) = \delta(t)$$

Take the Laplace transform of this equation:

$$sY(s) + Y(s) = 1$$

Solve for $Y(s)$:

$$Y(s) = \frac{1}{s+1}$$

Take inverse Laplace transform (by recognizing form of transform):

$$y(t) = e^{-t}u(t)$$

Solving Differential Equations with Laplace Transforms

Recognizing the form ...
Is there a more systematic way to take an inverse Laplace transform?

Yes ... and no.

Formally,

$$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st} ds$$

but this integral is not generally easy to compute.

This equation can be useful to prove theorems.

We will find better ways (e.g., partial fractions) to compute inverse transforms for common systems.

Solving Differential Equations with Laplace Transforms

Example 2:

$$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = \delta(t)$$

Laplace transform:

$$s^2Y(s) + 3sY(s) + 2Y(s) = 1$$

Solve:

$$Y(s) = \frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}$$

Inverse Laplace transform:

$$y(t) = \left(e^{-t} - e^{-2t}\right) u(t)$$

These forward and inverse Laplace transforms are easy if

- differential equation is linear with constant coefficients, and
- the input signal is an impulse function.

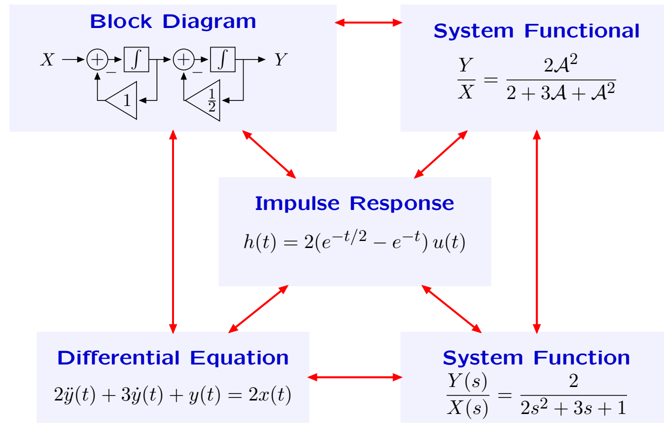
Properties of Laplace Transforms

The use of Laplace Transforms to solve differential equations depends on several important properties.

Property	$x(t)$	$X(s)$	ROC
Linearity	$ax_1(t)+bx_2(t)$	$aX_1(s)+bX_2(s)$	$\supset (R_1 \cap R_2)$
Delay by T	$x(t-T)$	$X(s)e^{-sT}$	R
Multiply by t	$tx(t)$	$-\frac{dX(s)}{ds}$	R
Multiply by $e^{-\alpha t}$	$x(t)e^{-\alpha t}$	$X(s+\alpha)$	shift R by $-\alpha$
Differentiate in t	$\frac{dx(t)}{dt}$	$sX(s)$	$\supset R$
Integrate in t	$\int_{-\infty}^{t} x(\tau)\,d\tau$	$\frac{X(s)}{s}$	$\supset \Big(R \cap \big(\mathrm{Re}(s)>0\big)\Big)$
Convolve in t	$\int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)\,d\tau$	$X_1(s)X_2(s)$	$\supset (R_1 \cap R_2)$

Concept Map: Continuous-Time Systems

Where does Laplace transform fit in?



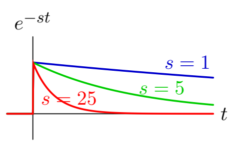
Initial Value Theorem

If $x(t)=0$ for $t<0$ and $x(t)$ contains no impulses or higher-order singularities at $t=0$ then

$$x(0^+) = \lim_{s\to\infty} sX(s)\,.$$

Consider $\lim_{s\to\infty} sX(s) = \lim_{s\to\infty} s\int_{-\infty}^{\infty} x(t)e^{-st}dt = \lim_{s\to\infty}\int_{0}^{\infty} x(t)\,se^{-st}dt.$

As $s\to\infty$ the function e^{-st} shrinks towards 0.



Area under e^{-st} is $\frac{1}{s}$ $\to$ area under se^{-st} is 1 $\to$ $\lim_{s\to\infty} se^{-st} = \delta(t)$!

$$\lim_{s\to\infty} sX(s) = \lim_{s\to\infty}\int_0^{\infty} x(t)se^{-st}dt \to \int_0^{\infty} x(t)\delta(t)dt = x(0^+)$$

(the 0^+ arises because the limit is from the right side.)

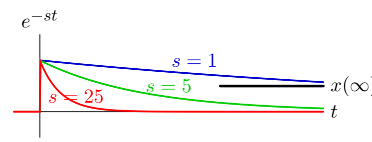
Final Value Theorem

If $x(t)=0$ for $t<0$ and $x(t)$ has a finite limit as $t\to\infty$

$$x(\infty) = \lim_{s\to 0} sX(s)\,.$$

Consider $\lim_{s\to 0} sX(s) = \lim_{s\to 0} s\int_{-\infty}^{\infty} x(t)e^{-st}dt = \lim_{s\to 0}\int_{0}^{\infty} x(t)\,se^{-st}dt.$

As $s\to 0$ the function e^{-st} flattens out. But again, the area under se^{-st} is always 1.



As $s\to 0$, area under se^{-st} monotonically shifts to higher values of t (e.g., the average value of se^{-st} is $\frac{1}{s}$ which grows as $s\to 0$).

In the limit, $\lim_{s\to 0} sX(s) \to x(\infty)\,.$