6.003: Signals and Systems

Relations between CT and DT:
Insights from Operators and Transforms

February 25, 2010

Mid-term Examination #1

Wednesday, March 3, 7:30-9:30pm, 34-101.

No recitations on the day of the exam.

Coverage:	Representations of CT and DT Systems
	Lectures 1–7
	Recitations 1–8
	Homeworks 1–4

Homework 4 will not collected or graded. Solutions will be posted.

Closed book: 1 page of notes ($8\frac{1}{2} \times 11$ inches; front and back).

Designed as 1-hour exam; two hours to complete.

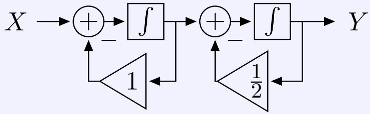
Review sessions during open office hours.

Conflict? Contact freeman@mit.edu before Friday, Feb. 26, 5pm.

Concept Map: Continuous-Time Systems

Relations among CT representations.

Block Diagram



System Functional

$$\frac{Y}{X} = \frac{2\mathcal{A}^2}{2 + 3\mathcal{A} + \mathcal{A}^2}$$

Impulse Response

$$h(t) = 2(e^{-t/2} - e^{-t})\,u(t)$$

Differential Equation

$$2\ddot{y}(t) + 3\dot{y}(t) + y(t) = 2x(t)$$

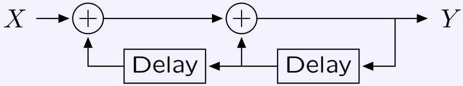
System Function

$$\frac{Y(s)}{X(s)} = \frac{2}{2s^2 + 3s + 1}$$

Concept Map: Discrete-Time Systems

Relations among DT representations.

Block Diagram



System Functional

$$\frac{Y}{X} = \frac{1}{1 - \mathcal{R} - \mathcal{R}^2}$$

Unit-Sample Response

$$h[n]: \ \ 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$$

Difference Equation

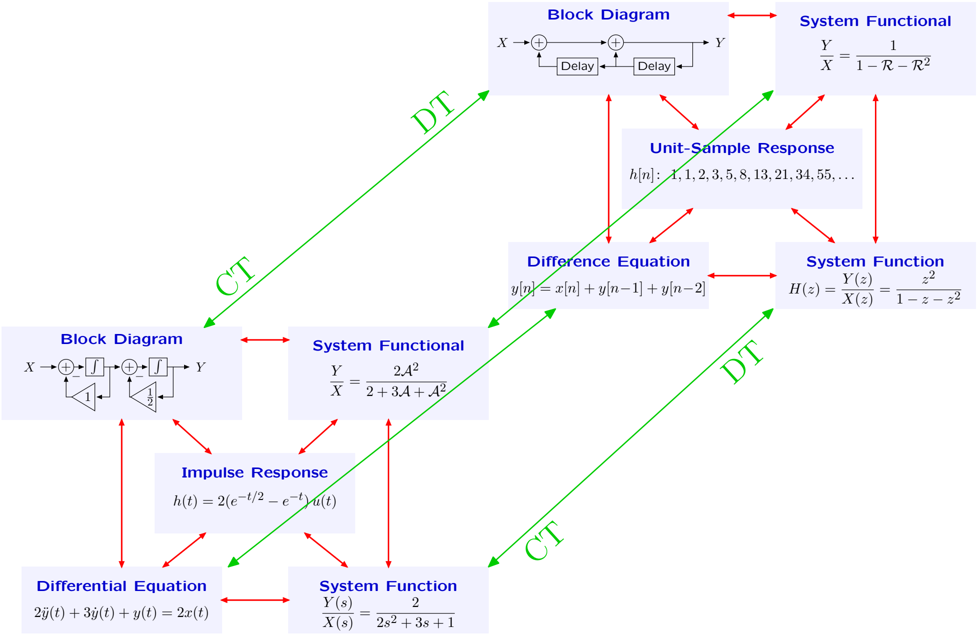
$$y[n] = x[n] + y[n-1] + y[n-2]$$

System Function

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^2}{1 - z - z^2}$$

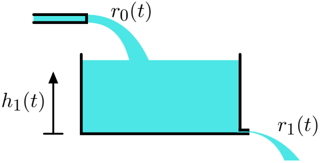
Concept Map

Relations between CT and DT representations.

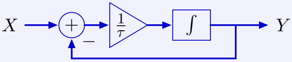


First-Order CT System

Example: leaky tank.



Block Diagram



System Functional

$$\frac{Y}{X} = \frac{\mathcal{A}}{\mathcal{A}+\tau}$$

Impulse Response

$$h(t) = \frac{1}{\tau}e^{-t/\tau}u(t)$$

Differential Equation

$$\tau \dot{r}_1(t) = r_0(t) - r_1(t)$$

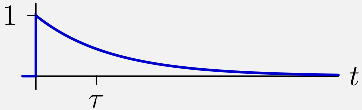
System Function

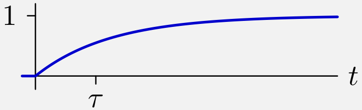
$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{1+\tau s}$$

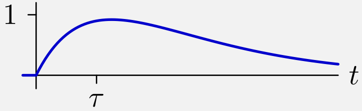
Check Yourself

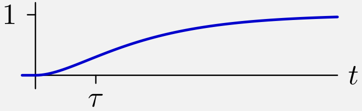
What is the "step response" of the leaky tank system?

$$u(t) \longrightarrow \boxed{\text{Leaky Tank}} \longrightarrow s(t) = ?$$

1. 

2. 

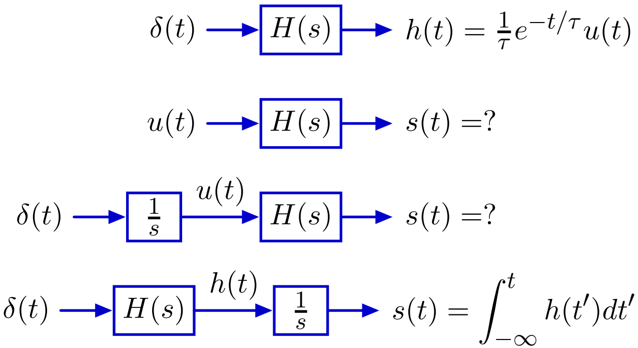
3. 

4. 

5. none of the above

Check Yourself

What is the "step response" of the leaky tank system?



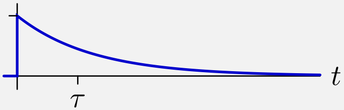
$$s(t) = \int_{-\infty}^{t} \frac{1}{\tau} e^{-t'/\tau} u(t')dt' = \int_{0}^{t} \frac{1}{\tau} e^{-t'/\tau} dt' = \left(1 - e^{-t/\tau}\right) u(t)$$

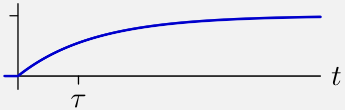
Reasoning with systems.

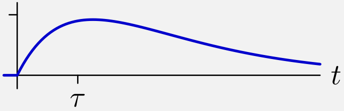
Check Yourself

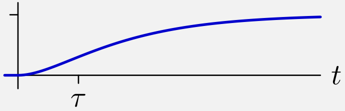
What is the "step response" of the leaky tank system? 2

$$u(t) \longrightarrow \boxed{\text{Leaky Tank}} \longrightarrow s(t) = ?$$

1. 

2. 

3. 

4. 

5. none of the above

Forward Euler Approximation

Approximate leaky-tank response using forward Euler approach.

Substitute

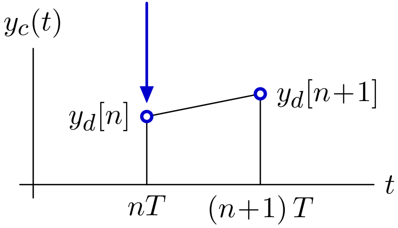
$$x_d[n] = x_c(nT)$$

$$y_d[n] = y_c(nT)$$

$$\dot{y}_c(nT) \approx \frac{y_c\big((n+1)T\big) - y_c\big(nT\big)}{T} = \frac{y_d[n+1] - y_d[n]}{T}$$

$$y_c(nT) = y_d[n]$$

$$\dot{y}_c(nT) = \frac{y_d[n{+}1] - y_d[n]}{T}$$



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Approximate leaky-tank response using forward Euler approach.

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$$y_d[n] = y_c(nT)$$

$$\dot{y}_c(nT) \approx \frac{y_c\big((n+1)T\big) - y_c\big(nT\big)}{T} = \frac{y_d[n+1] - y_d[n]}{T}$$

into the differential equation

$$\tau \dot{y}_c(t) = x_c(t) - y_c(t)$$

to obtain

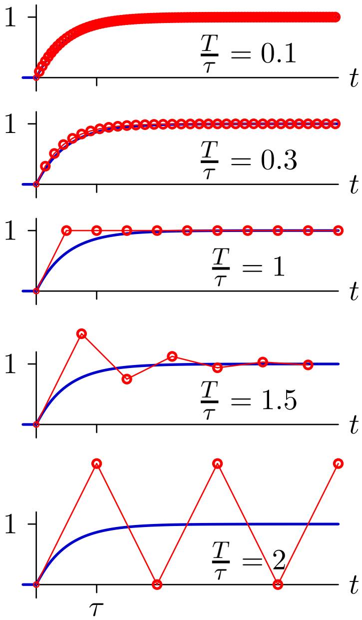
$$\frac{\tau}{T}\Big(y_d[n+1] - y_d[n]\Big) = x_d[n] - y_d[n]\,.$$

Solve:

$$y_d[n+1] - \left(1 - \frac{T}{\tau}\right) y_d[n] = \frac{T}{\tau} x_d[n]$$

Forward Euler Approximation

Plot.



Why is this approximation badly behaved?

Check Yourself

DT approximation:

$$y_d[n+1] - \left(1 - \frac{T}{\tau}\right) y_d[n] = \frac{T}{\tau} x_d[n]$$

Find the DT pole.

1. $z = \frac{T}{\tau}$
2. $z = 1 - \frac{T}{\tau}$
3. $z = \frac{\tau}{T}$
4. $z = -\frac{\tau}{T}$
5. $z = \frac{1}{1 + \frac{T}{\tau}}$

Check Yourself

DT approximation:

$$y_d[n+1] - \left(1 - \frac{T}{\tau}\right) y_d[n] = \frac{T}{\tau} x_d[n]$$

Take the Z transform:

$$zY_d(z) - \left(1 - \frac{T}{\tau}\right) Y_d(z) = \frac{T}{\tau} X_d(z)$$

Solve for the system function:

$$H(z) = \frac{Y_d(z)}{X_d(z)} = \frac{\frac{T}{\tau}}{z - \left(1 - \frac{T}{\tau}\right)}$$

Pole at $z = 1 - \dfrac{T}{\tau}$.

Check Yourself

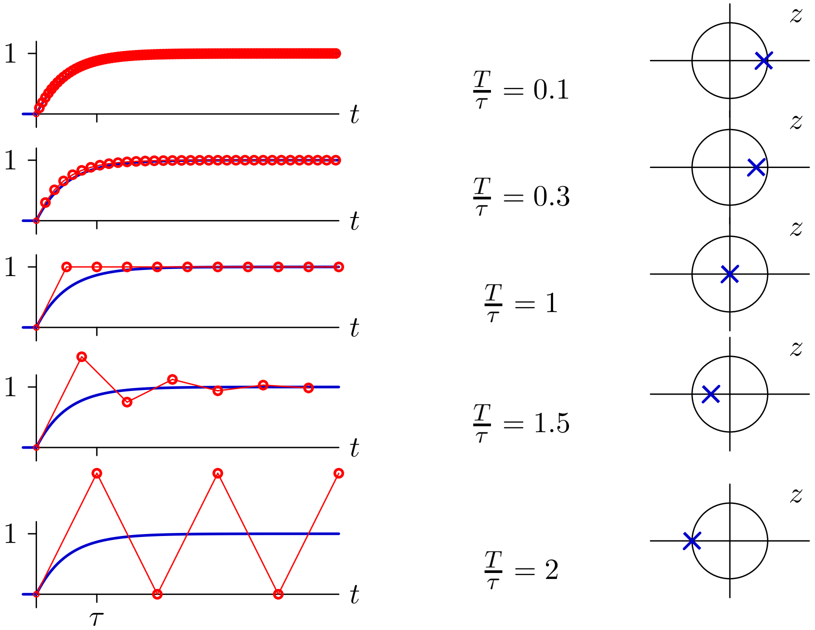
DT approximation:

$$y_d[n+1] - \left(1 - \frac{T}{\tau}\right) y_d[n] = \frac{T}{\tau} x_d[n]$$

Find the DT pole. 2

1. $z = \frac{T}{\tau}$
2. $z = 1 - \frac{T}{\tau}$
3. $z = \frac{\tau}{T}$
4. $z = -\frac{\tau}{T}$
5. $z = \frac{1}{1 + \frac{T}{\tau}}$

Dependence of DT pole on Stepsize



The CT pole was fixed ($s = -\frac{1}{\tau}$). Why is the DT pole changing?

Dependence of DT pole on Stepsize

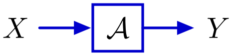
Change in DT pole: problem specific or property of forward Euler?

Dependence of DT pole on Stepsize

Change in DT pole: problem specific or property of forward Euler?

Approach: make a systems model of forward Euler method.

CT block diagrams: adders, gains, and integrators:

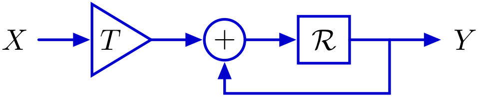


$$\dot{y}(t) = x(t)$$

Forward Euler approximation:

$$\frac{y[n+1] - y[n]}{T} = x[n]$$

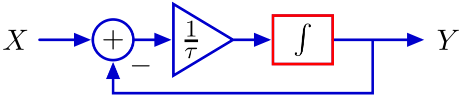
Equivalent system:



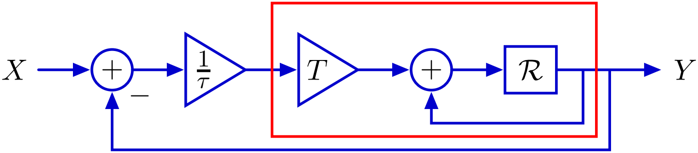
Forward Euler: substitute equivalent system for all integrators.

Example: leaky tank system

Started with leaky tank system:



Replace integrator with forward Euler rule:



Write system functional:

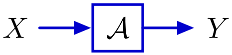
$$\frac{Y}{X}=\frac{\frac{T}{\tau}\frac{\mathcal{R}}{1-\mathcal{R}}}{1+\frac{T}{\tau}\frac{\mathcal{R}}{1-\mathcal{R}}}=\frac{\frac{T}{\tau}\mathcal{R}}{1-\mathcal{R}+\frac{T}{\tau}\mathcal{R}}=\frac{\frac{T}{\tau}\mathcal{R}}{1-\left(1-\frac{T}{\tau}\right)\mathcal{R}}$$

Equivalent to system we previously developed:

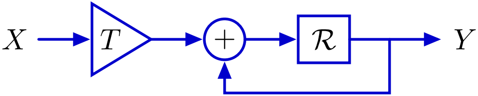
$$y_d[n+1]-\left(1-\frac{T}{\tau}\right)y_d[n]=\frac{T}{\tau}x_d[n]$$

Model of Forward Euler Method

Replace every integrator in the CT system



with the forward Euler model:



Substitute the DT operator for $\mathcal{A}$:

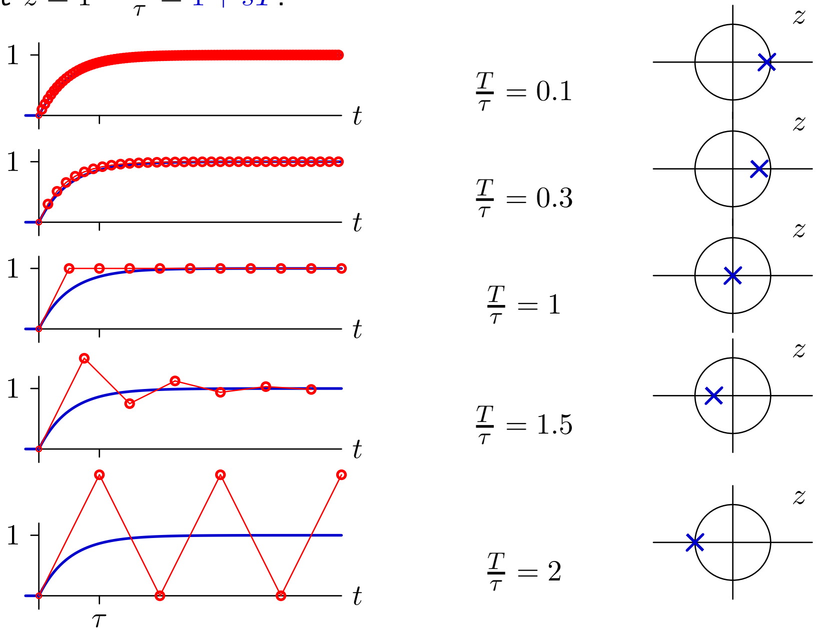
$$\mathcal{A} = \frac{1}{s} \rightarrow \frac{T\mathcal{R}}{1-\mathcal{R}} = \frac{\frac{T}{z}}{1-\frac{1}{z}} = \frac{T}{z-1}$$

Forward Euler maps $s \rightarrow \dfrac{z-1}{T}$.

Or equivalently: $z = 1 + sT$.

Dependence of DT pole on Stepsize

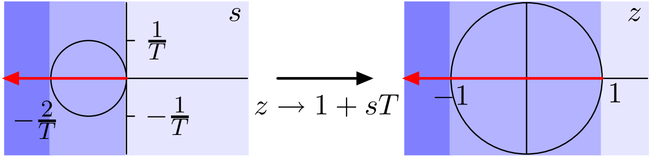
Pole at $z = 1 - \frac{T}{\tau} = 1 + sT$.



Forward Euler: Mapping CT poles to DT poles

Forward Euler Map:

$$
\begin{array}{ccc}
s & \rightarrow & z = 1 + sT \\
0 & & 1 \\
-\frac{1}{T} & & 0 \\
-\frac{2}{T} & & -1
\end{array}
$$



DT stability: CT pole must be inside circle of radius $\frac{1}{T}$ at $s = -\frac{1}{T}$.

$$-\frac{2}{T} < -\frac{1}{\tau} < 0 \quad \rightarrow \quad \frac{T}{\tau} < 2$$

Backward Euler Approximation

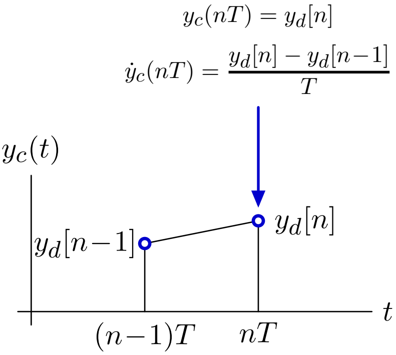
We can do a similar analysis of the backward Euler method.

Substitute

$$x_d[n] = x_c(nT)$$

$$y_d[n] = y_c(nT)$$

$$\dot{y}_c(nT) \approx \frac{y_c\big(nT\big) - y_c\big((n-1)T\big)}{T} = \frac{y_d[n] - y_d[n-1]}{T}$$



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into the differential equation

$$\tau \dot{y}_c(t) = x_c(t) - y_c(t)$$

to obtain

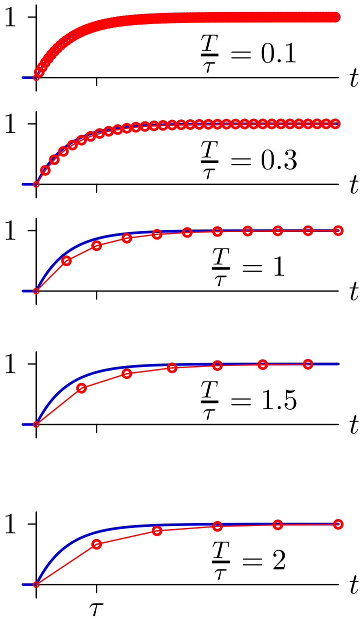
$$\frac{\tau}{T}\Big(y_d[n] - y_d[n-1]\Big) = x_d[n] - y_d[n]\,.$$

Solve:

$$\left(1 + \frac{T}{\tau}\right) y_d[n] - y_d[n-1] = \frac{T}{\tau} x_d[n]$$

Backward Euler Approximation

Plot.



This approximation is better behaved. Why?

Check Yourself

DT approximation:

$$\left(1+\frac{T}{\tau}\right)y_d[n]-y_d[n-1]=\frac{T}{\tau}x_d[n]$$

Find the DT pole.

1. $z=\frac{T}{\tau}$
2. $z=1-\frac{T}{\tau}$
3. $z=\frac{\tau}{T}$
4. $z=-\frac{\tau}{T}$
5. $z=\frac{1}{1+\frac{T}{\tau}}$

Check Yourself

DT approximation:

$$\left(1+\frac{T}{\tau}\right)y_d[n]-y_d[n-1]=\frac{T}{\tau}x_d[n]$$

Take the Z transform:

$$\left(1+\frac{T}{\tau}\right)Y_d(z)-z^{-1}Y_d(z)=\frac{T}{\tau}X_d(z)$$

Find the system function:

$$H(z)=\frac{Y_d(z)}{X_d(z)}=\frac{\frac{T}{\tau}z}{\left(1+\frac{T}{\tau}\right)z-1}$$

Pole at $z=\dfrac{1}{1+\frac{T}{\tau}}$.

Check Yourself

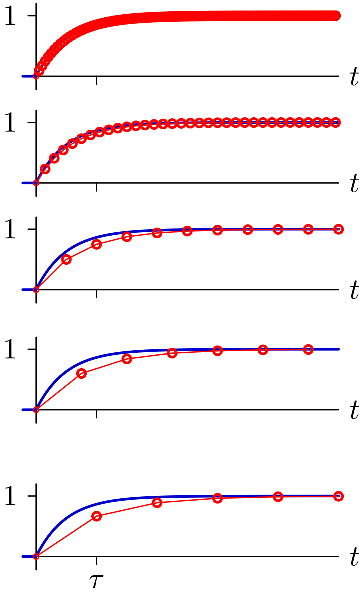
DT approximation:

$$y_d[n+1] - \left(1 - \frac{T}{\tau}\right) y_d[n] = \frac{T}{\tau} x_d[n]$$

Find the DT pole. 5

1. $z = \dfrac{T}{\tau}$
2. $z = 1 - \dfrac{T}{\tau}$
3. $z = \dfrac{\tau}{T}$
4. $z = -\dfrac{\tau}{T}$
5. $z = \dfrac{1}{1 + \frac{T}{\tau}}$

Dependence of DT pole on Stepsize



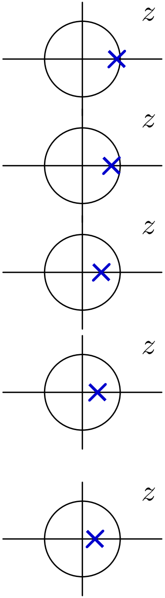
$$\frac{T}{\tau} = 0.1$$

$$\frac{T}{\tau} = 0.3$$

$$\frac{T}{\tau} = 1$$

$$\frac{T}{\tau} = 1.5$$

$$\frac{T}{\tau} = 2$$

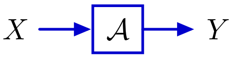


Why is this approximation better behaved?

Dependence of DT pole on Stepsize

Make a systems model of backward Euler method.

CT block diagrams: adders, gains, and integrators:

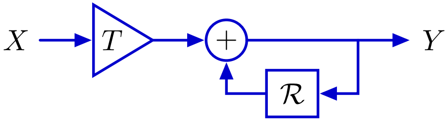


$$\dot{y}(t) = x(t)$$

Backward Euler approximation:

$$\frac{y[n] - y[n-1]}{T} = x[n]$$

Equivalent system:

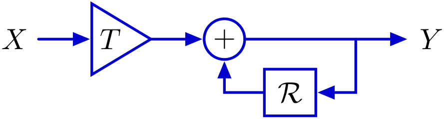


Backward Euler: substitute equivalent system for all integrators.

Model of Backward Euler Method

Replace every integrator in the CT system

with the backward Euler model:



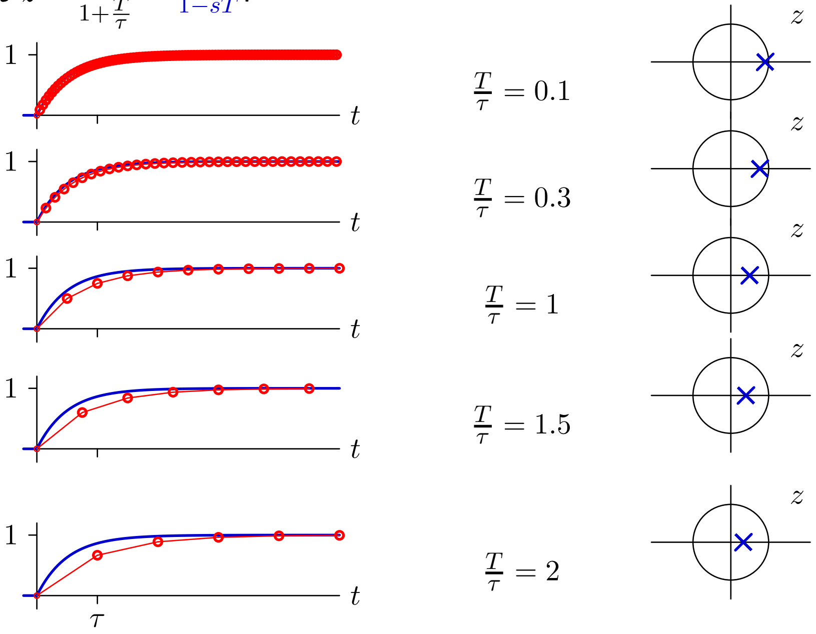
Substitute the DT operator for $\mathcal{A}$:

$$\mathcal{A} = \frac{1}{s} \rightarrow \frac{T}{1-\mathcal{R}} = \frac{T}{1-\frac{1}{z}}$$

Backward Euler maps $z \rightarrow \dfrac{1}{1-sT}$.

Dependence of DT pole on Stepsize

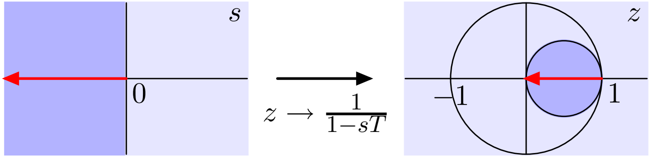
Pole at $z = \frac{1}{1+\frac{T}{\tau}} = \frac{1}{1-sT}$.



Backward Euler: Mapping CT poles to DT poles

Backward Euler Map:

s	$\rightarrow$	$z = \frac{1}{1-sT}$
0		1
$-\frac{1}{T}$		$\frac{1}{2}$
$-\frac{2}{T}$		$\frac{1}{3}$



$$z \rightarrow \frac{1}{1-sT}$$

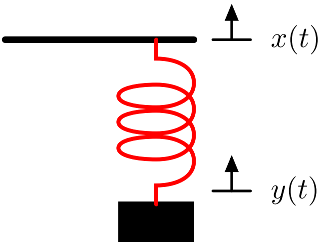
The entire left half-plane maps inside a circle with radius $\frac{1}{2}$ at $z = \frac{1}{2}$.

If CT system is stable, then DT system is also stable.

Masses and Springs, Forwards and Backwards

In Homework 2, you investigated three numerical approximations to a mass and spring system:

- forward Euler
- backward Euler
- centered method



Trapezoidal Rule

The trapezoidal rule uses centered differences.

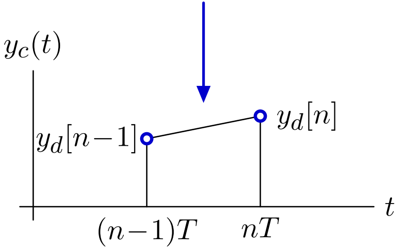
$$\dot{y}(t) = x(t)$$

Trapezoidal rule:

$$\frac{y[n] - y[n-1]}{T} = \frac{x[n] + x[n-1]}{2}$$

$$y_c\left((n+\tfrac{1}{2})T\right) = \frac{y_d[n] + y_d[n-1]}{2}$$

$$\dot{y}_c\left((n+\tfrac{1}{2})T\right) = \frac{y_d[n] - y_d[n-1]}{T}$$



Trapezoidal Rule

The trapezoidal rule uses centered differences.

$$\dot{y}(t) = x(t)$$

Trapezoidal rule:

$$\frac{y[n] - y[n-1]}{T} = \frac{x[n] + x[n-1]}{2}$$

Z transform:

$$H(z) = \frac{Y(s)}{X(s)} = \frac{T}{2}\left(\frac{1+z^{-1}}{1-z^{-1}}\right) = \frac{T}{2}\left(\frac{z+1}{z-1}\right)$$

Map:

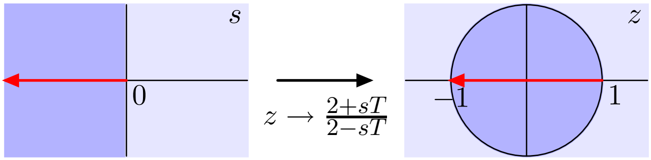
$$\mathcal{A} = \frac{1}{s} \rightarrow \frac{T}{2}\left(\frac{z+1}{z-1}\right)$$

Trapezoidal rule maps $z \rightarrow \dfrac{1+\frac{sT}{2}}{1-\frac{sT}{2}}$.

Trapezoidal Rule: Mapping CT poles to DT poles

Trapezoidal Map:

$$
\begin{array}{ccc}
s & \rightarrow & z = \frac{1+\frac{sT}{2}}{1-\frac{sT}{2}} \\
0 & & 1 \\
-\frac{1}{T} & & \frac{1}{3} \\
-\frac{2}{T} & & 0 \\
-\infty & & -1 \\
j\omega & & \frac{2+j\omega T}{2-j\omega T}
\end{array}
$$

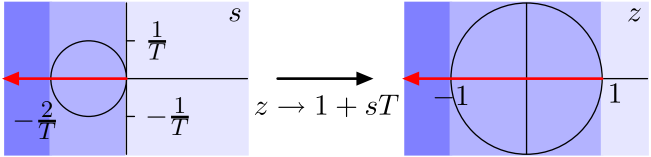


The entire left-half plane maps inside the unit circle.

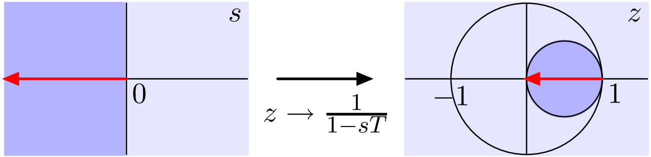
The $j\omega$ axis maps onto the unit circle

Mapping s to z: Leaky-Tank System

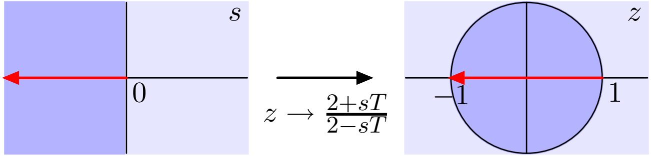
Forward Euler Method



Backward Euler Method

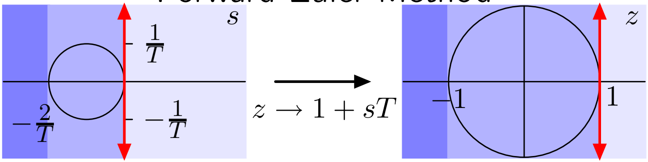


Trapezoidal Rule

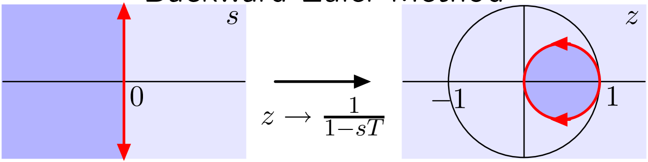


Mapping s to z: Mass and Spring System

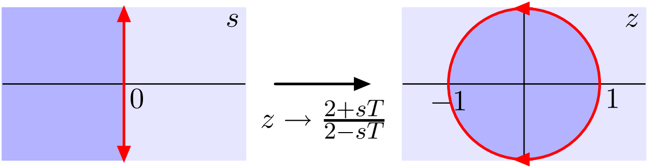
Forward Euler Method



Backward Euler Method

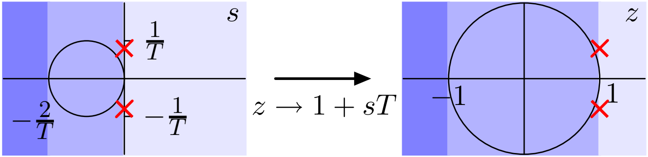


Trapezoidal Rule

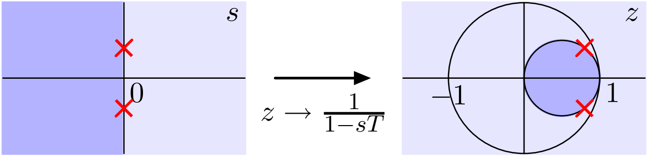


Mapping s to z: Mass and Spring System

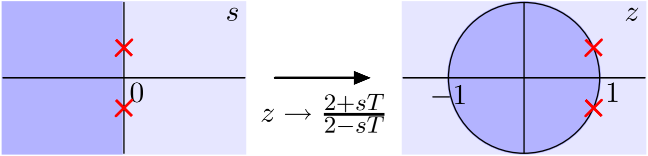
Forward Euler Method



Backward Euler Method



Trapezoidal Rule



Concept Map

Relations between CT and DT representations.