6.003: Signals and Systems

Convolution

March 2, 2010

Mid-term Examination #1

Tomorrow, Wednesday, March 3, 7:30-9:30pm, 34-101.

No recitations tomorrow.

Coverage: Representations of CT and DT Systems
Lectures 1–7
Recitations 1–8
Homeworks 1–4

Homework 4 will not collected or graded. Solutions are posted.

Closed book: 1 page of notes ($8\frac{1}{2} \times 11$ inches; front and back).

Designed as 1-hour exam; two hours to complete.

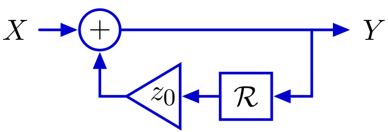
Multiple Representations of CT and DT Systems

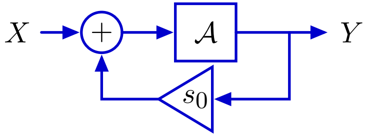
Verbal descriptions: preserve the rationale.

Difference/differential equations: mathematically compact.

$$y[n] = x[n] + z_0\, y[n-1] \qquad\qquad \dot{y}(t) = x(t) + s_0\, y(t)$$

Block diagrams: illustrate signal flow paths.





Operator representations: analyze systems as polynomials.

$$\frac{Y}{X} = \frac{1}{1 - z_0\mathcal{R}} \qquad\qquad \frac{Y}{X} = \frac{\mathcal{A}}{1 - s_0\mathcal{A}}$$

Transforms: representing diff. equations with algebraic equations.

$$H(z) = \frac{z}{z - z_0} \qquad\qquad H(s) = \frac{1}{s - s_0}$$

Convolution

Representing a system by a single signal.

Responses to arbitrary signals

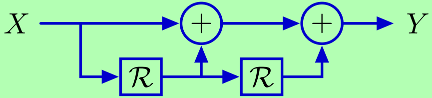
Although we have focused on responses to simple signals ($\delta[n], \delta(t)$) we are generally interested in responses to more complicated signals.

How do we compute responses to a more complicated input signals?

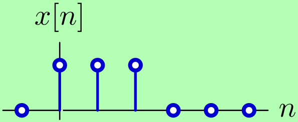
No problem for difference equations / block diagrams.
$\rightarrow$ use step-by-step analysis.

Check Yourself

Example: Find $y[3]$



when the input is

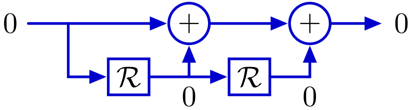


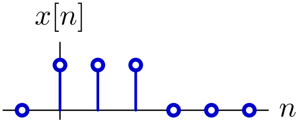
1. 1 2. 2 3. 3 4. 4 5. 5

0. none of the above

Responses to arbitrary signals

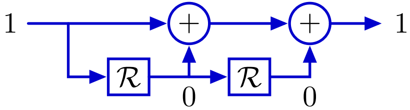
Example.





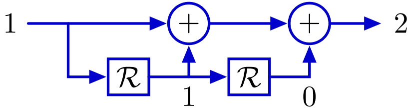
Responses to arbitrary signals

Example.



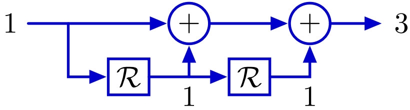
Responses to arbitrary signals

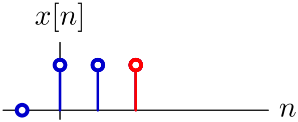
Example.

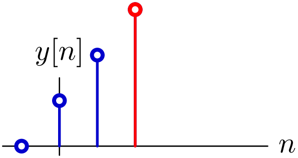


Responses to arbitrary signals

Example.

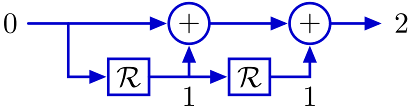


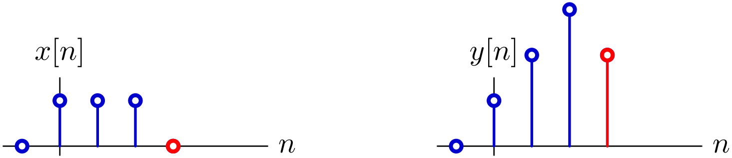




Responses to arbitrary signals

Example.





Responses to arbitrary signals

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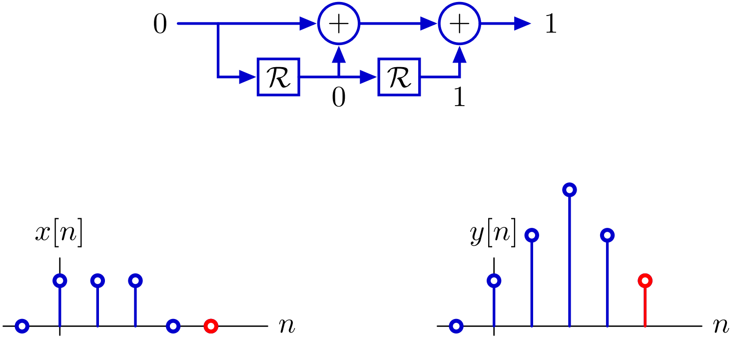

Responses to arbitrary signals

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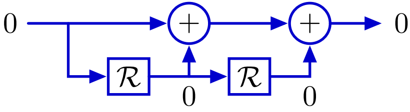
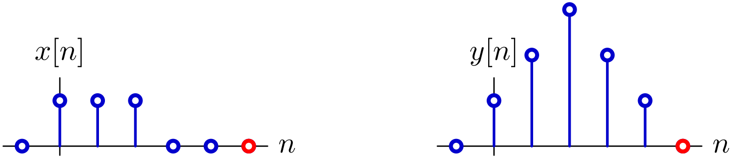
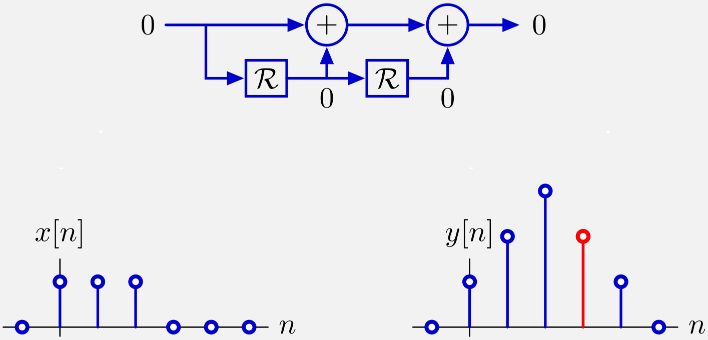




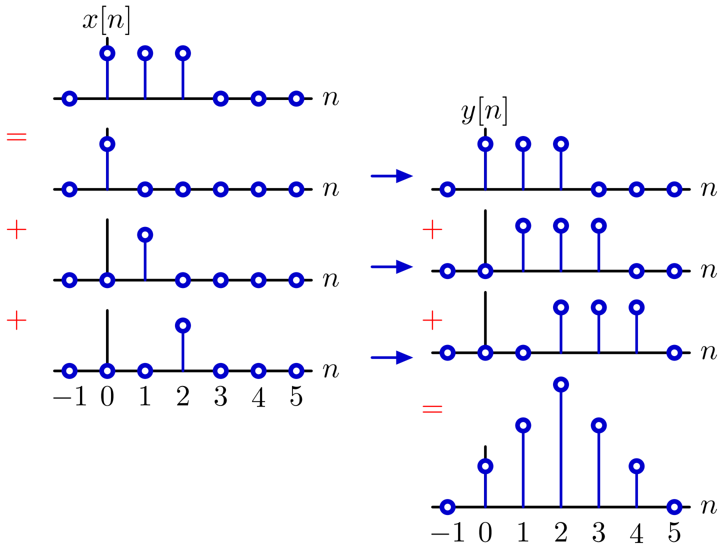
Check Yourself

What is $y[3]$? 2



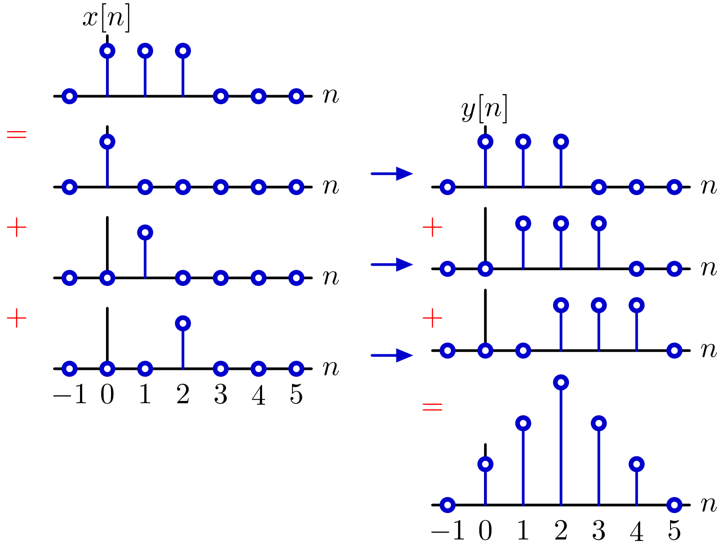
Alternative: Superposition

Break input into additive parts and sum the responses to the parts.



Superposition

Break input into additive parts and sum the responses to the parts.



Superposition works if the system is **linear**.

Linearity

A system is linear if its response to a weighted sum of inputs is equal to the weighted sum of its responses to each of the inputs.

Given

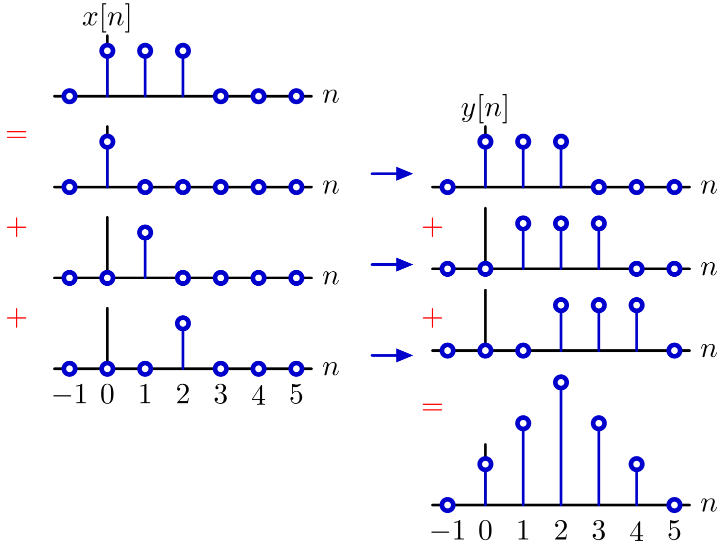
and

the system is linear if

is true for all α and β.

Superposition

Break input into additive parts and sum the responses to the parts.



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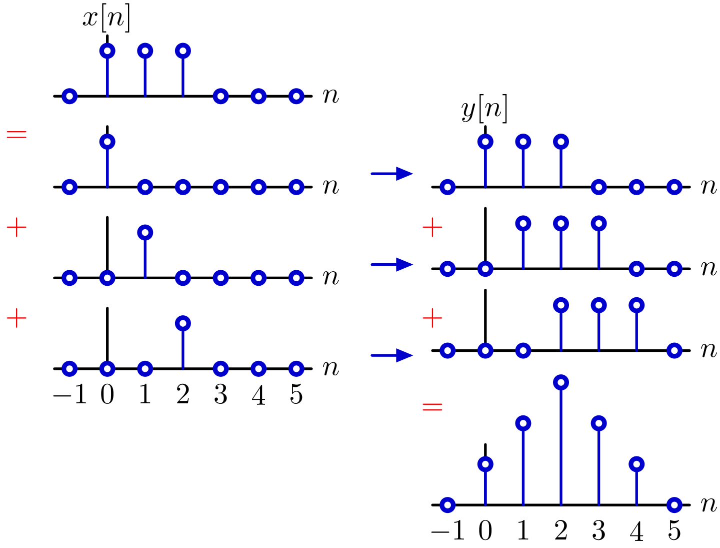
Superposition

Break input into additive parts and sum the responses to the parts.



Reponses to parts are easy to compute if system is **time-invariant**.

Time-Invariance

A system is time-invariant if delaying the input to the system simply delays the output by the same amount of time.

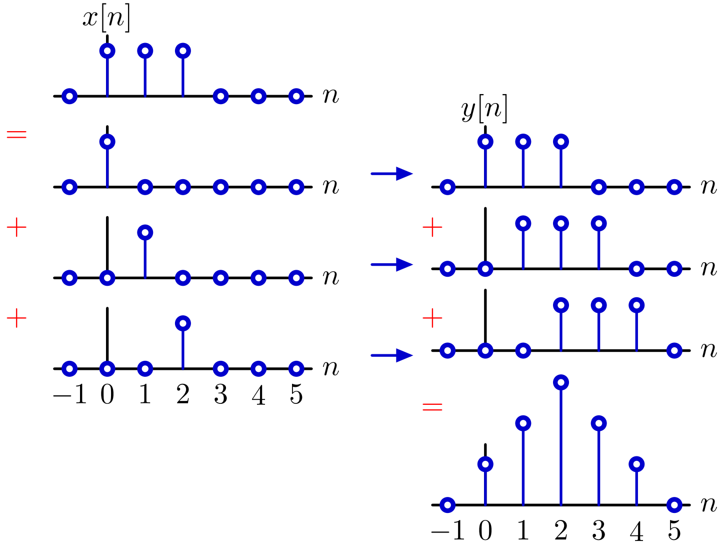
Given

the system is time invariant if

is true for all n_0.

Superposition

Break input into additive parts and sum the responses to the parts.



Superposition is easy if the system is **linear** and **time-invariant**.

Structure of Superposition

If a system is linear and time-invariant (LTI) then its output is the sum of weighted and shifted unit-sample responses.

$$\delta[n] \longrightarrow \boxed{\text{system}} \longrightarrow h[n]$$

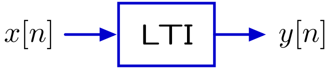
$$\delta[n-k] \longrightarrow \boxed{\text{system}} \longrightarrow h[n-k]$$

$$x[k]\delta[n-k] \longrightarrow \boxed{\text{system}} \longrightarrow x[k]h[n-k]$$

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \longrightarrow \boxed{\text{system}} \longrightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Convolution

Response of an LTI system to an arbitrary input.



$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \equiv (x * h)[n]$$

This operation is called **convolution**.

Notation

Convolution is represented with an asterisk.

$$\sum_{k=-\infty}^{\infty} x[k]h[n-k] \equiv (x*h)[n]$$

It is customary (but confusing) to abbreviate this notation:

$$(x*h)[n] = x[n]*h[n]$$

Notation

Do not be fooled by the confusing notation.

Confusing (but conventional) notation:

$$\sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n] * h[n]$$

$x[n] * h[n]$ looks like an operation of samples; but it is not!

$$x[1] * h[1] \neq (x * h)[1]$$

Convolution operates on signals not samples.

Unambiguous notation:

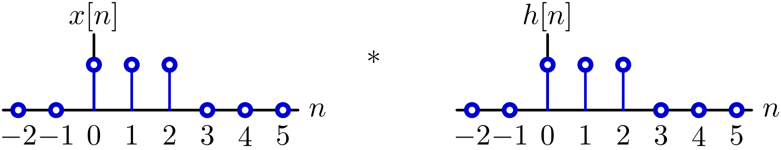
$$\sum_{k=-\infty}^{\infty} x[k]h[n-k] \equiv (x * h)[n]$$

The symbols x and h represent DT signals.

Convolving x with h generates a new DT signal $x * h$.

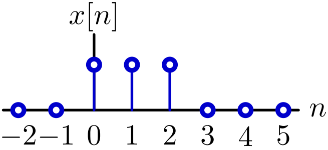
Structure of Convolution

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$



Structure of Convolution

$$y[0] = \sum_{k=-\infty}^{\infty} x[k]h[0-k]$$



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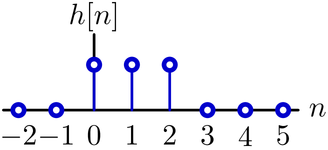

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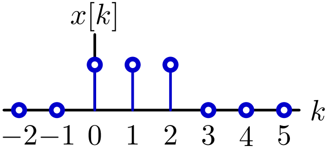

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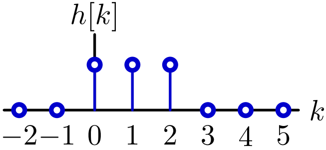
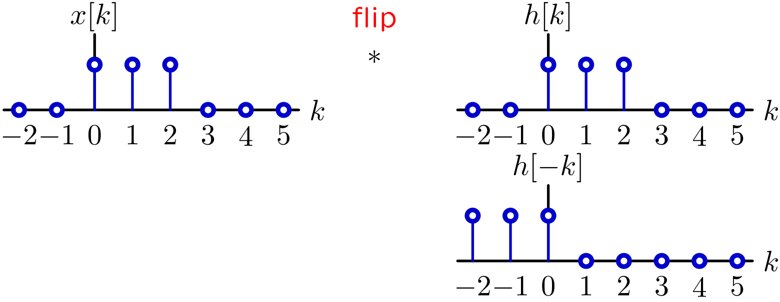


Structure of Convolution

$$y[0] = \sum_{k=-\infty}^{\infty} x[k]h[0 - k]$$



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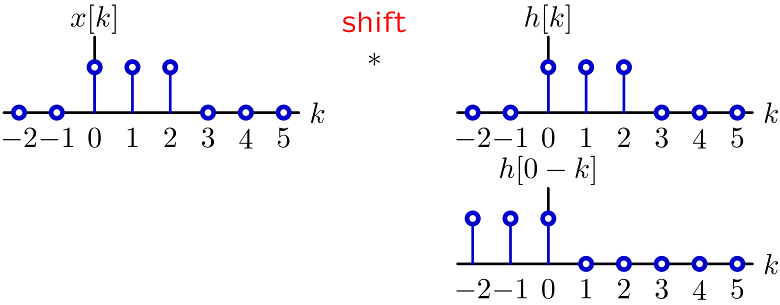

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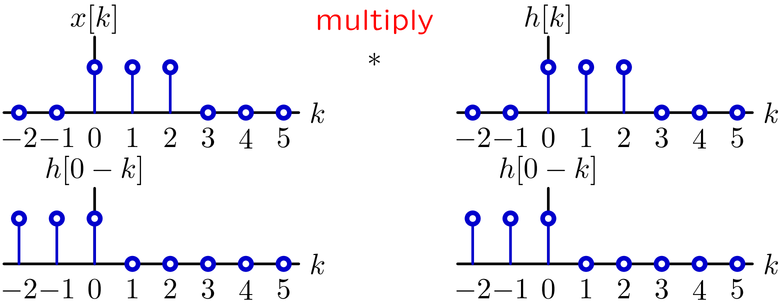

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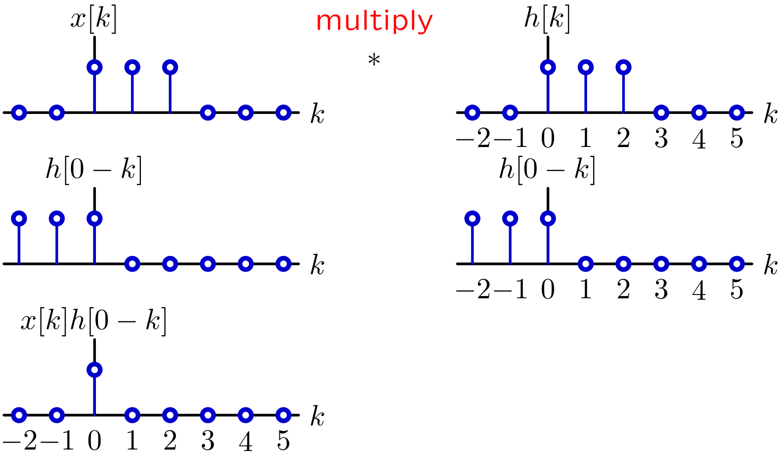

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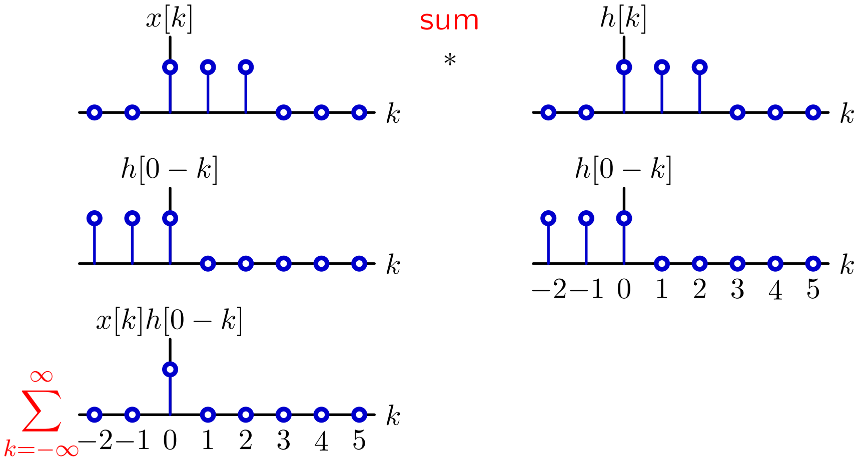

Structure of Convolution

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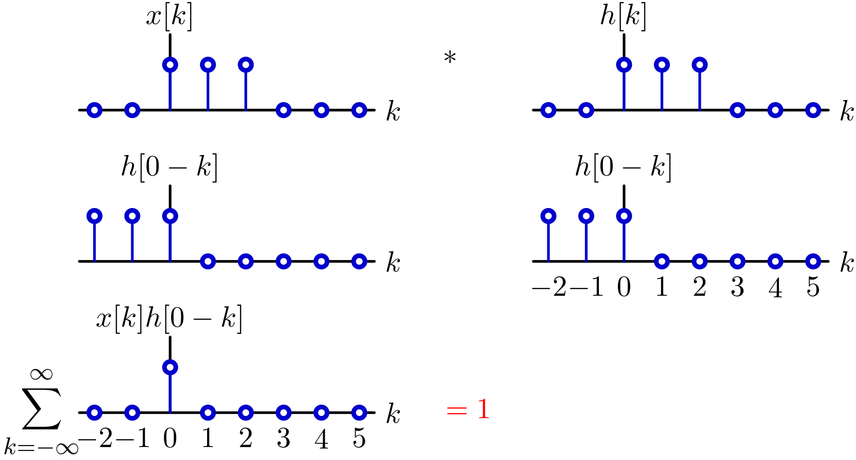
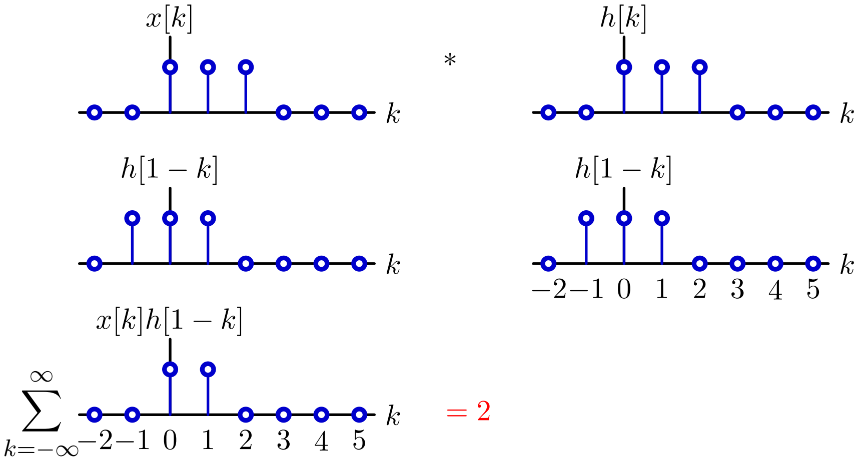


$= 1$

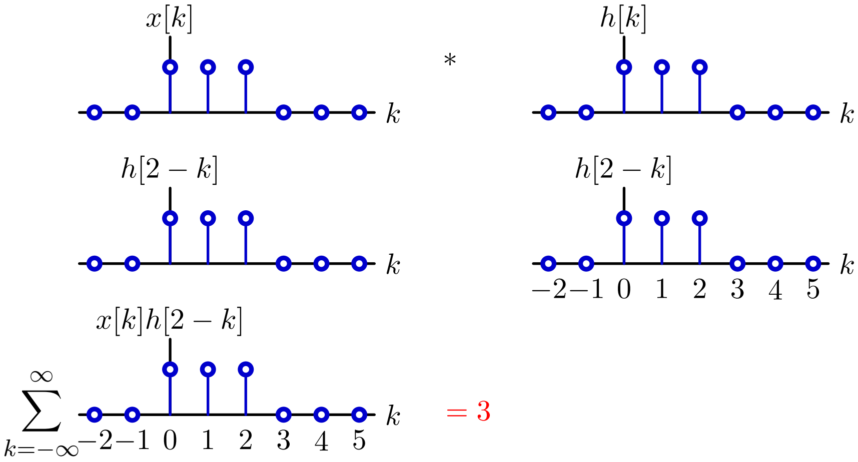
Structure of Convolution

$$y[1] = \sum_{k=-\infty}^{\infty} x[k]h[1-k]$$



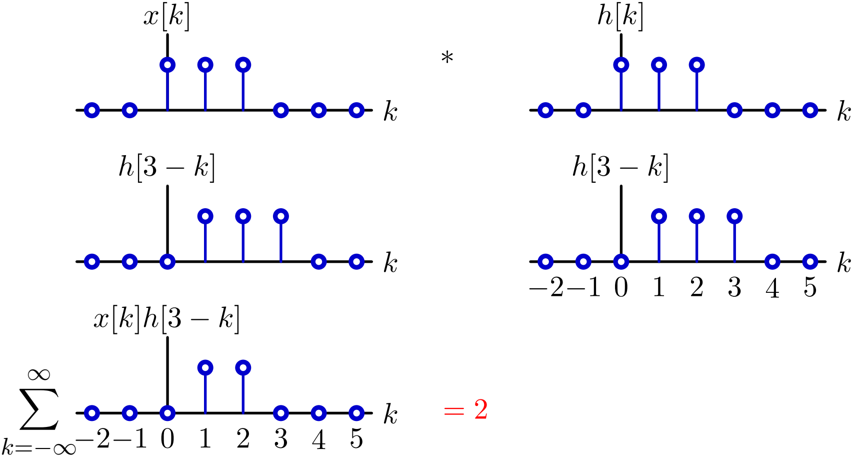
Structure of Convolution

$$y[2] = \sum_{k=-\infty}^{\infty} x[k]h[2-k]$$



Structure of Convolution

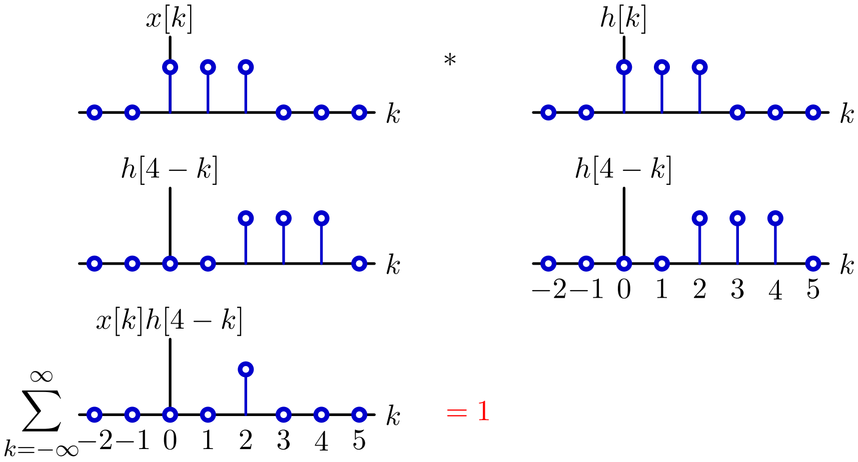
$$y[3] = \sum_{k=-\infty}^{\infty} x[k]h[3-k]$$



$$\sum_{k=-\infty}^{\infty} x[k]h[3-k] = 2$$

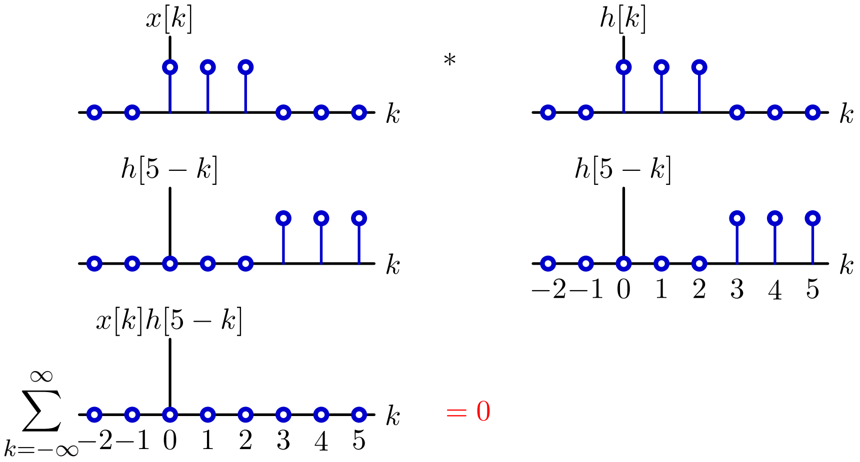
Structure of Convolution

$$y[4] = \sum_{k=-\infty}^{\infty} x[k]h[4-k]$$

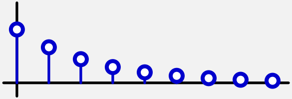


Structure of Convolution

$$y[{\color{red}5}] = \sum_{k=-\infty}^{\infty} x[k]h[5-k]$$



Check Yourself



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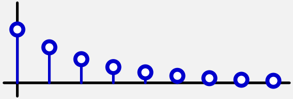
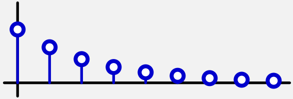
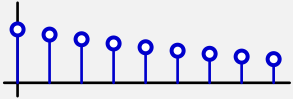
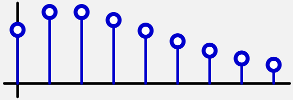
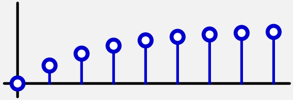


Which plot shows the result of the convolution above?

1. 

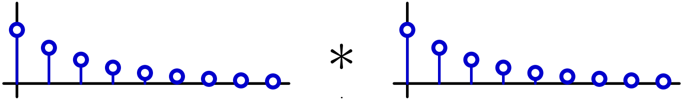
2. 

3. 

4. 

5. none of the above

Check Yourself



Express mathematically:

$$\left(\left(\frac{2}{3}\right)^n u[n]\right) * \left(\left(\frac{2}{3}\right)^n u[n]\right) = \sum_{k=-\infty}^{\infty} \left(\left(\frac{2}{3}\right)^k u[k]\right) \times \left(\left(\frac{2}{3}\right)^{n-k} u[n-k]\right)$$

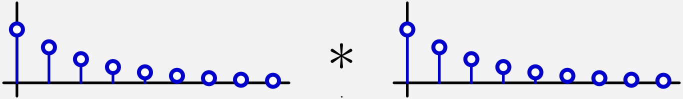
$$= \sum_{k=0}^{n} \left(\frac{2}{3}\right)^k \times \left(\frac{2}{3}\right)^{n-k}$$

$$= \sum_{k=0}^{n} \left(\frac{2}{3}\right)^n = \left(\frac{2}{3}\right)^n \sum_{k=0}^{n} 1$$

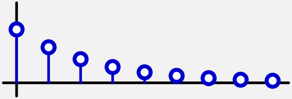
$$= (n+1)\left(\frac{2}{3}\right)^n u[n]$$

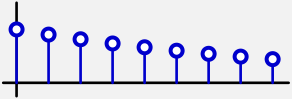
$$= 1,\ \frac{4}{3},\ \frac{4}{3},\ \frac{32}{27},\ \frac{80}{81},\ \ldots$$

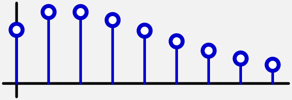
Check Yourself

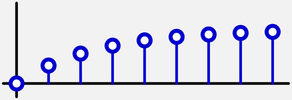


Which plot shows the result of the convolution above? 3

1. 

2. 

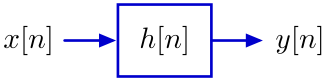
3. 

4. 

5. none of the above

Convolution

Representing an LTI system by a single signal.



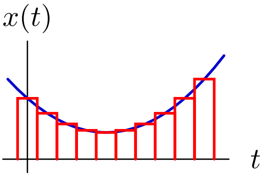
Unit-sample response $h[n]$ is a complete description of an LTI system.

Given $h[n]$ one can compute the response $y[n]$ to any arbitrary input signal $x[n]$:

$$y[n] = (x * h)[n] \equiv \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

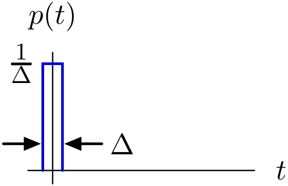
CT Convolution

The same sort of reasoning applies to CT signals.



$$x(t) = \lim_{\Delta \to 0} \sum_k x(k\Delta)p(t - k\Delta)\Delta$$

where

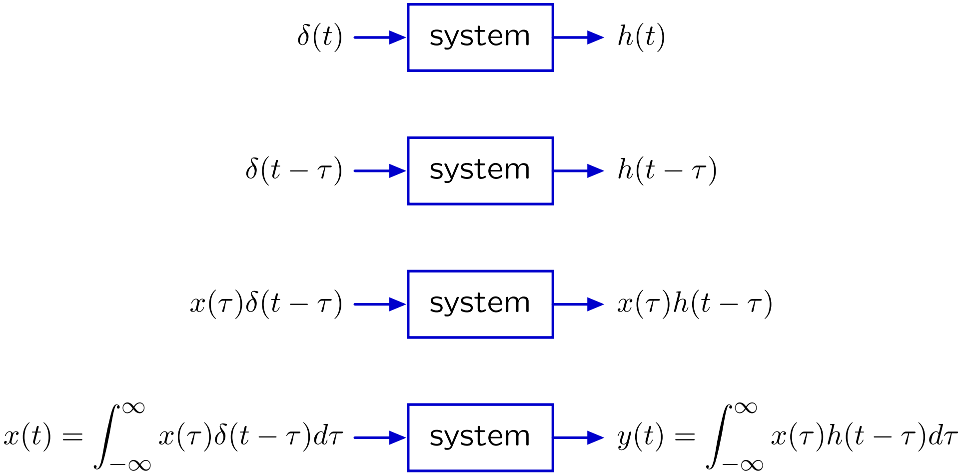


As $\Delta \to 0$, $k\Delta \to \tau$, $\Delta \to d\tau$, and $p(t) \to \delta(t)$:

$$x(t) \to \int_{-\infty}^{\infty} x(\tau)\delta(t - \tau)d\tau$$

Structure of Superposition

If a system is linear and time-invariant (LTI) then its output is the integral of weighted and shifted unit-impulse responses.



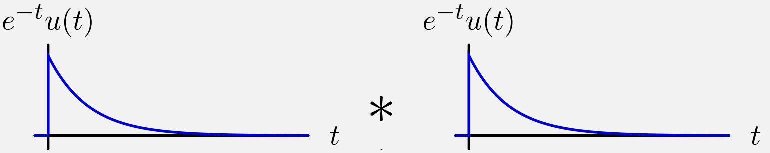
CT Convolution

Convolution of CT signals is analogous to convolution of DT signals.

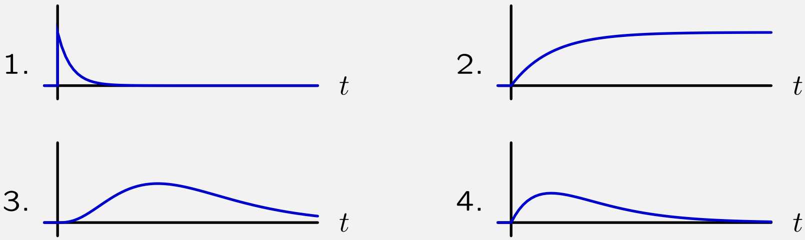
DT: $$y[n] = (x * h)[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

CT: $$y(t) = (x * h)(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Check Yourself



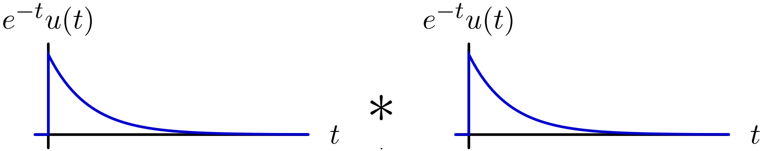
Which plot shows the result of the convolution above?



5. none of the above

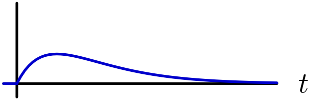
Check Yourself

Which plot shows the result of the following convolution?

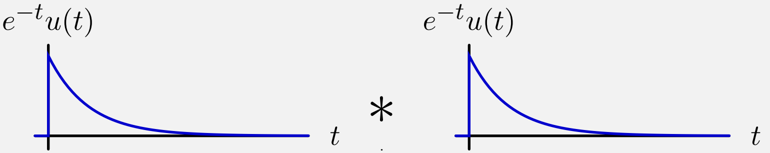


$$\left(e^{-t}u(t)\right) * \left(e^{-t}u(t)\right) = \int_{-\infty}^{\infty} e^{-\tau}u(\tau)e^{-(t-\tau)}u(t-\tau)d\tau$$

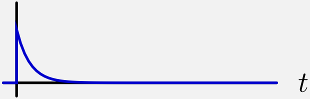
$$= \int_{0}^{t} e^{-\tau}e^{-(t-\tau)}d\tau = e^{-t}\int_{0}^{t} d\tau = te^{-t}u(t)$$

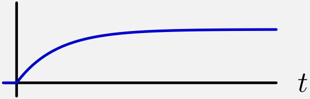


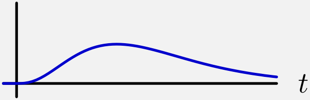
Check Yourself

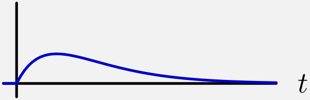


Which plot shows the result of the convolution above? 4

1. 

2. 

3. 

4. 

5. none of the above

Convolution

Convolution is an important **computational tool.**

Example: characterizing LTI systems

- Determine the unit-sample response $h[n]$.
- Calculate the output for an arbitrary input using convolution:

$$y[n] = (x * h)[n] = \sum x[k]h[n-k]$$

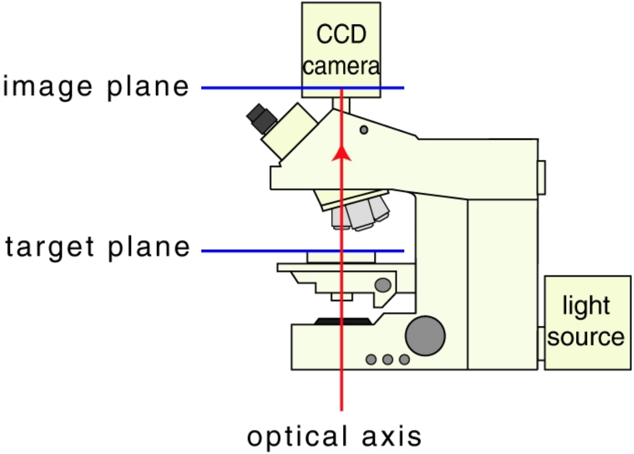
Applications of Convolution

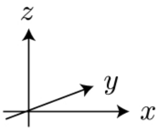
Convolution is an important **conceptual tool:** it provides an important new way to **think** about the behaviors of systems.

Example systems: microscopes and telescopes.

Microscope

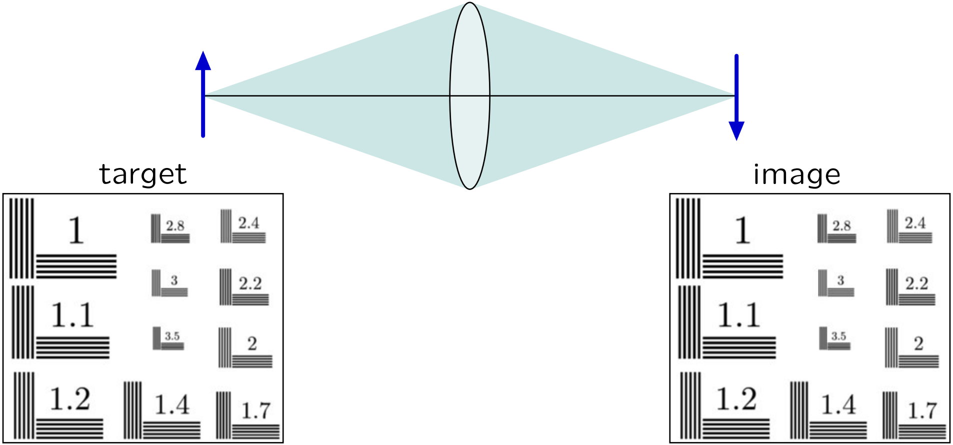
Images from even the best microscopes are blurred.





Microscope

A perfect lens transforms a spherical wave of light from the target into a spherical wave that converges to the image.



Blurring is inversely related to the diameter of the lens.

Microscope

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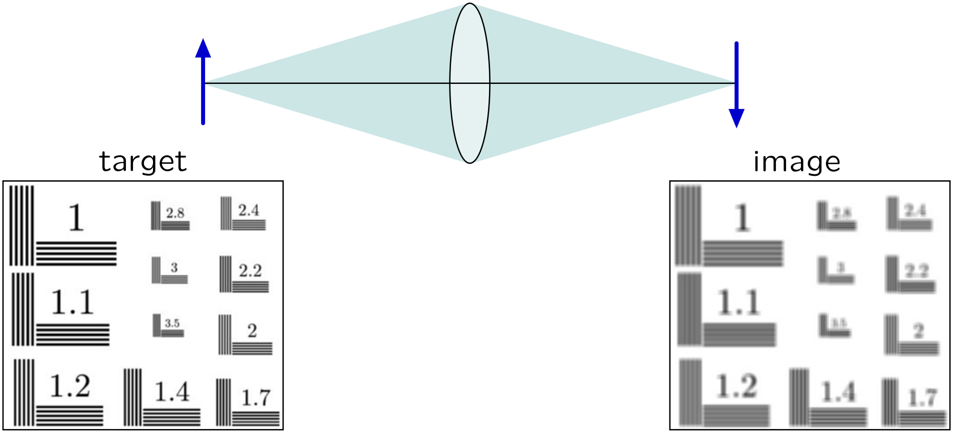

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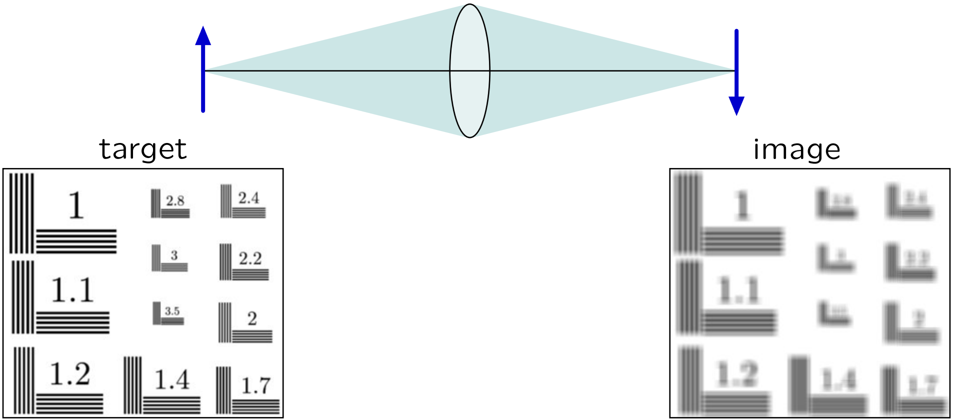
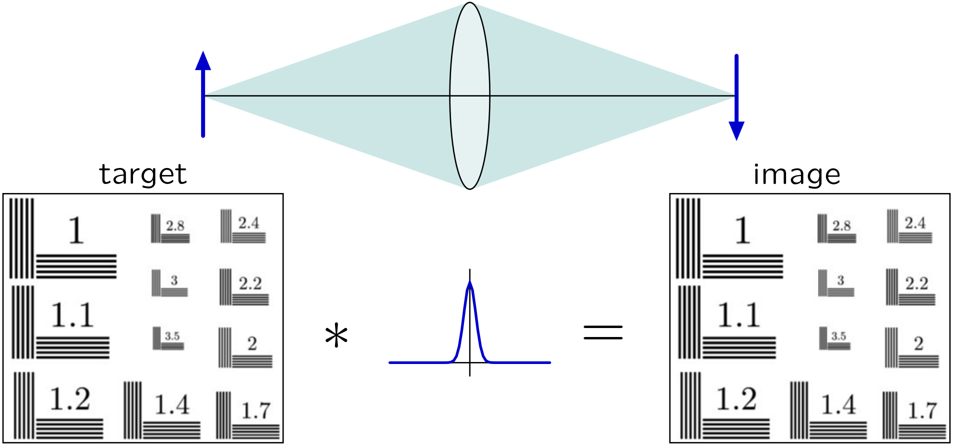


Blurring is inversely related to the diameter of the lens.

Microscope

Blurring can be represented by convolving the image with the optical "point-spread-function" (3D impulse response).



Blurring is inversely related to the diameter of the lens.

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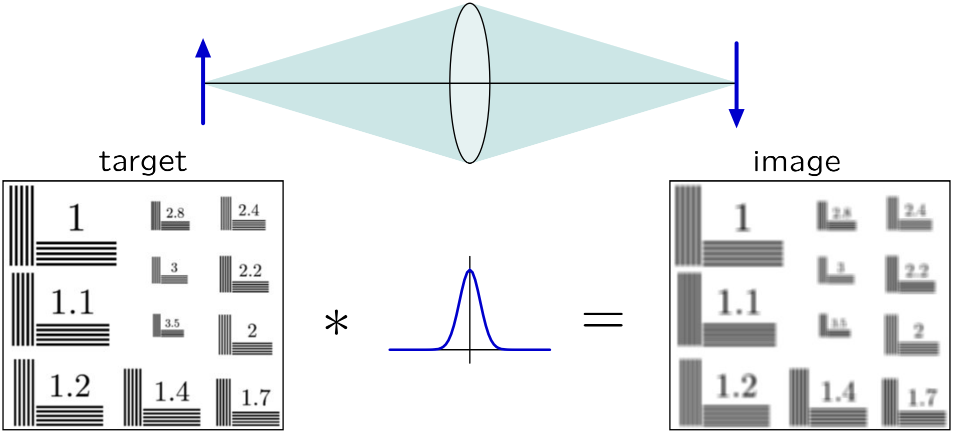

Blurring is inversely related to the diameter of the lens.

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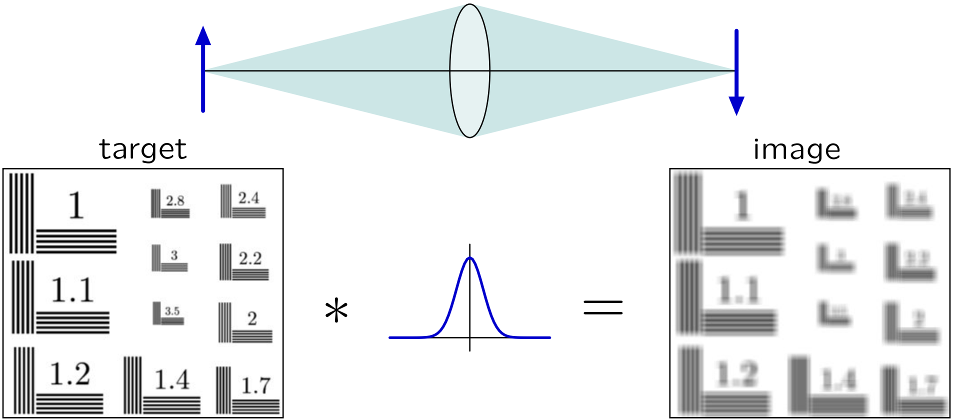
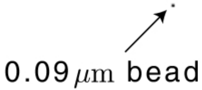


Blurring is inversely related to the diameter of the lens.

Microscope

Measuring the "impulse response" of a microscope.

Image diameter ≈ 6 times target diameter: target $\rightarrow$ impulse.



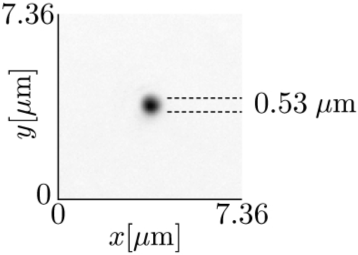
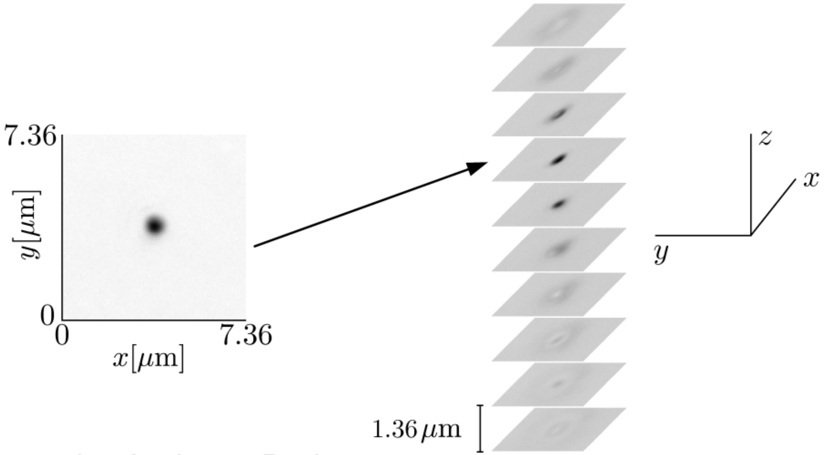


image of a 0.09 μm bead

images by Anthony Patire

Microscope

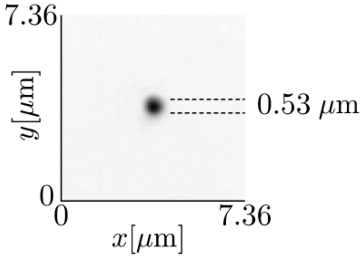
Images at different focal planes can be assembled to form a three-dimensional impulse response (point-spread function).

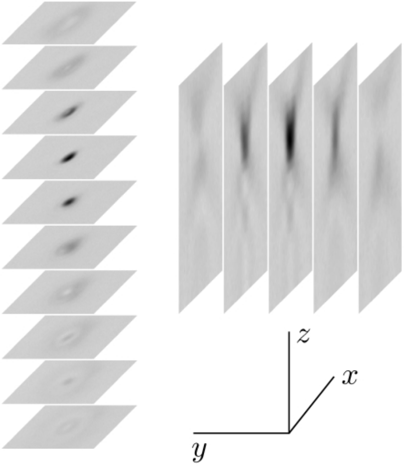


images by Anthony Patire

Microscope

Blurring along the optical axis is better visualized by resampling the three-dimensional impulse response.

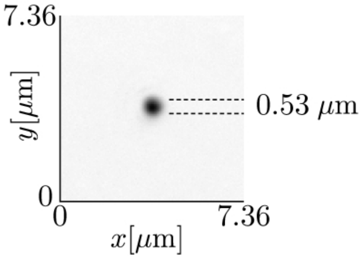


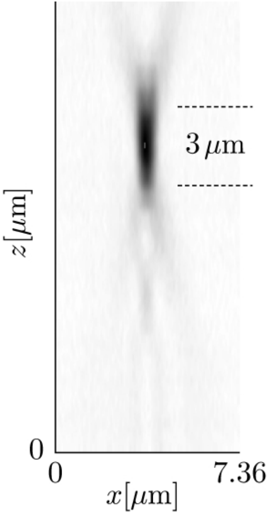


images by Anthony Patire

Microscope

Blurring is much greater along the optical axis than it is across the optical axis.





images by Anthony Patire

Microscope

The point-spread function (3D impulse response) is a useful way to characterize a microscope. It provides a direct measure of blurring, which is an important figure of merit for optics.

Hubble Space Telescope

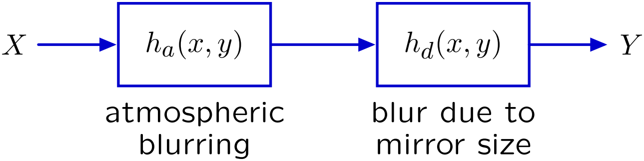
Hubble Space Telescope (1990-)

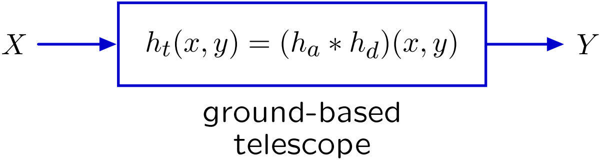
http://hubblesite.org

Hubble Space Telescope

Why build a space telescope?

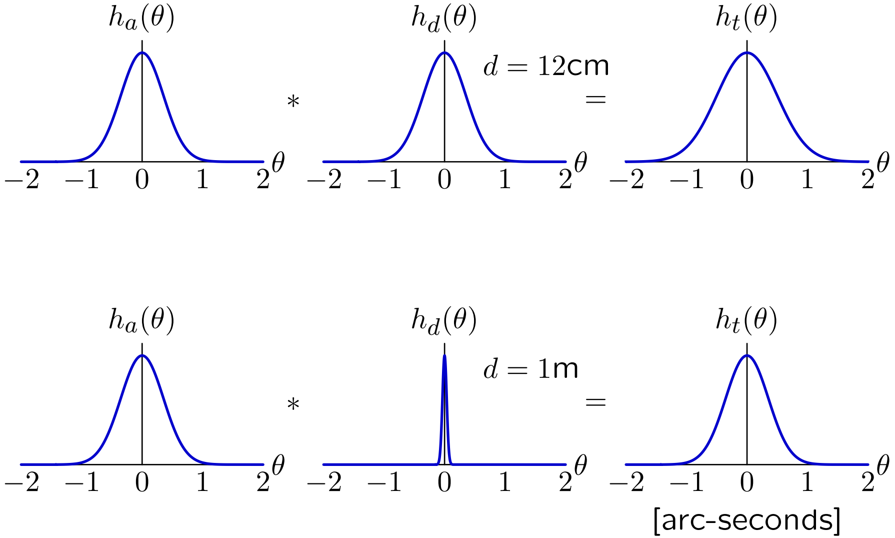
Telescope images are blurred by the telescope lenses AND by atmospheric turbulence.





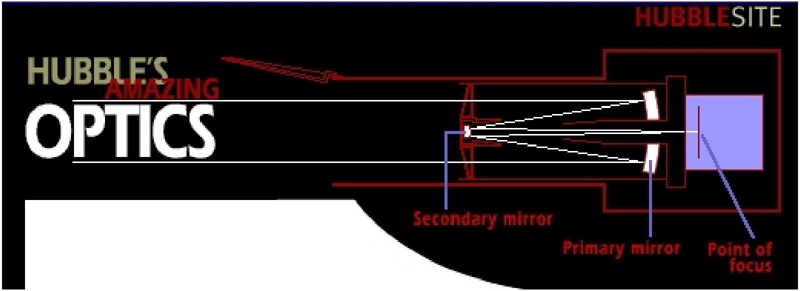
Hubble Space Telescope

Telescope blur can be respresented by the convolution of blur due to atmospheric turbulence and blur due to mirror size.



Hubble Space Telescope

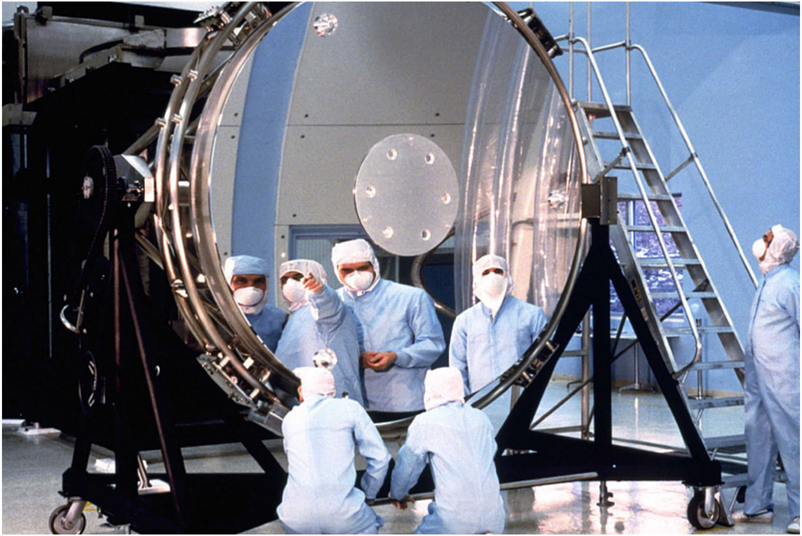
The main optical components of the Hubble Space Telescope are two mirrors.



http://hubblesite.org

Hubble Space Telescope

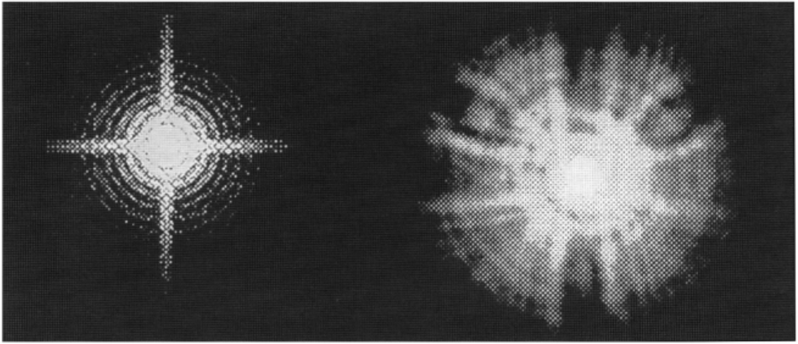
The diameter of the primary mirror is 2.4 meters.



http://hubblesite.org

Hubble Space Telescope

Hubble's first pictures of distant stars (May 20, 1990) were more blurred than expected.



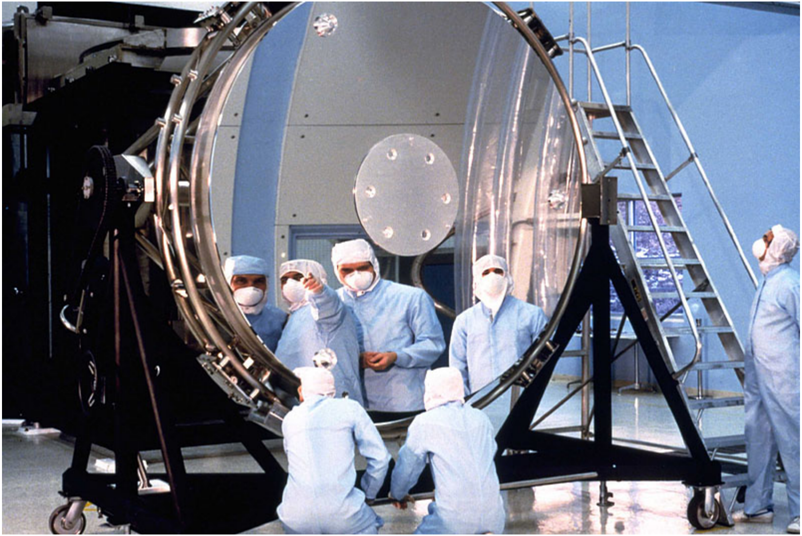
expected point-spread function

early Hubble image of distant star

http://hubblesite.org

Hubble Space Telescope

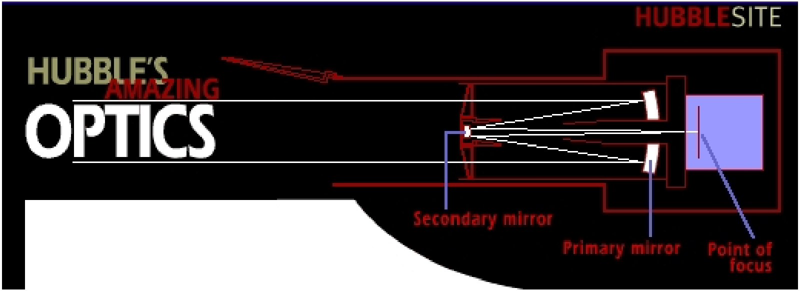
The parabolic mirror was ground 4 μm too flat!

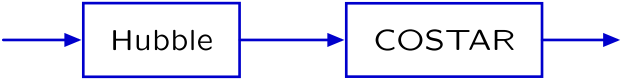


http://hubblesite.org

Hubble Space Telescope

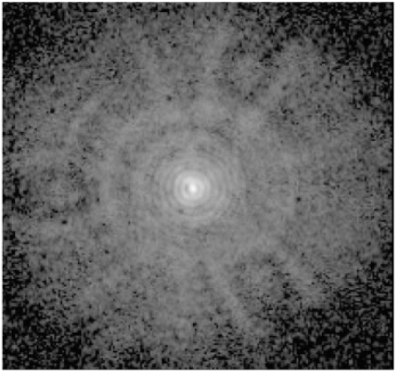
Corrective Optics Space Telescope Axial Replacement (COSTAR): eyeglasses for Hubble!



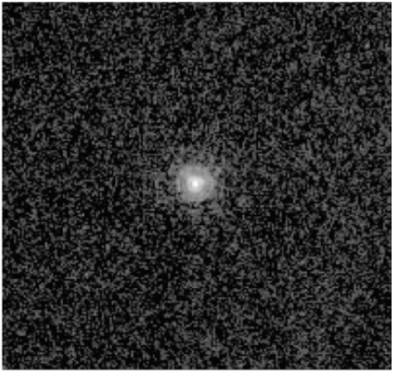


Hubble Space Telescope

Hubble images before and after COSTAR.



before

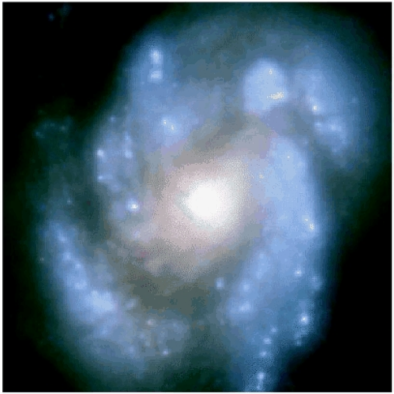


after

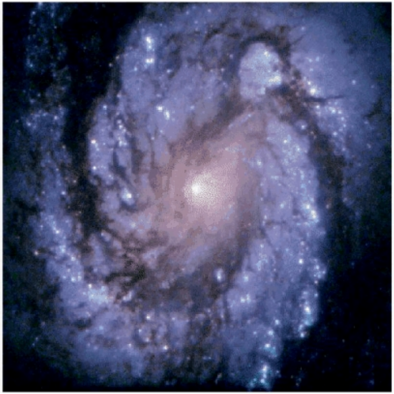
http://hubblesite.org

Hubble Space Telescope

Hubble images before and after COSTAR.



before

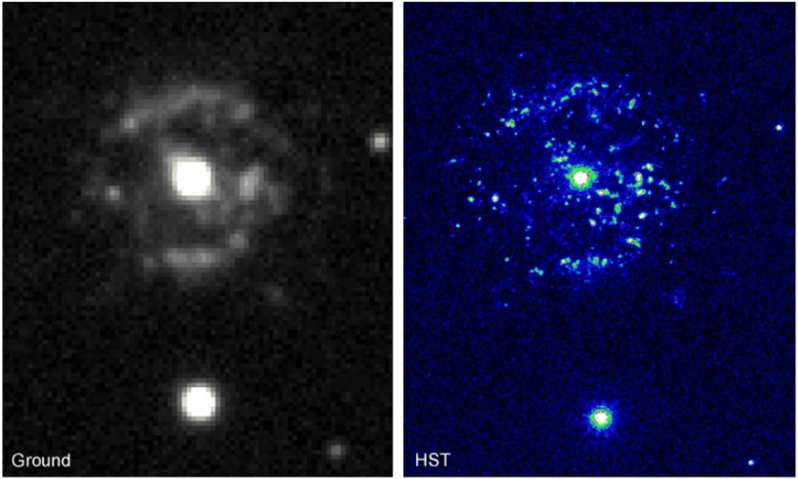


after

http://hubblesite.org

Hubble Space Telescope

Images from ground-based telescope and Hubble.



http://hubblesite.org

Impulse Response: Summary

The impulse response is a complete description of a linear, time-invariant system.

One can find the output of such a system by convolving the input signal with the impulse response.

The impulse response is an especially useful description of some types of systems, e.g., optical systems, where blurring is an important figure of merit.