6.003: Signals and Systems

CT Fourier Transform

April 8, 2010

CT Fourier Transform

Representing signals by their frequency content.

$$X(j\omega)= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \qquad \text{("analysis" equation)}$$

$$x(t)= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \qquad \text{("synthesis" equation)}$$

- generalizes Fourier series to represent aperiodic signals.
- equals Laplace transform $X(s)|_{s=j\omega}$ if ROC includes $j\omega$ axis.
  $\rightarrow$ inherits properties of Laplace transform.
- complex-valued function of **real** domain ω.
- simple "inverse" relation
  $\rightarrow$ more general than table-lookup method for inverse Laplace.
  $\rightarrow$ "duality."
- **filtering**.
- **applications in physics**.

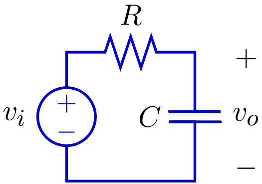
Filtering

Notion of a filter.

LTI systems

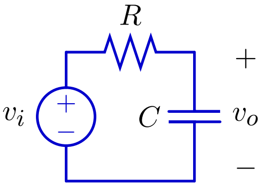
- cannot create new frequencies.
- can only scale magnitudes and shift phases of existing components.

Example: Low-Pass Filtering with an RC circuit

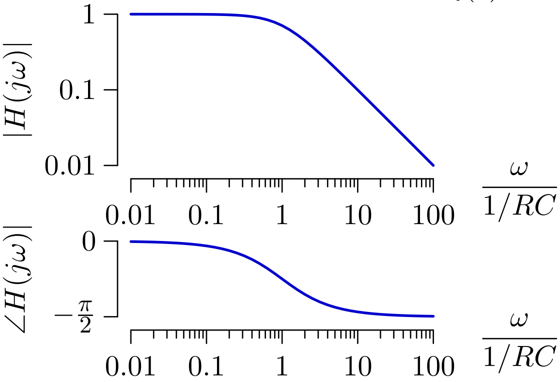


Lowpass Filter

Calculate the frequency response of an RC circuit.

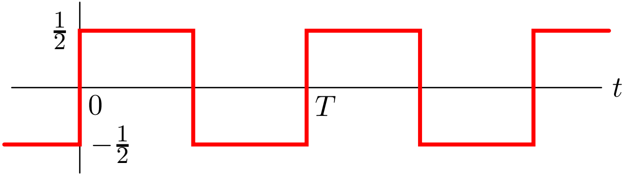


$$
\begin{aligned}
\text{KVL:} \quad v_i(t) &= Ri(t) + v_o(t) \\
\text{C:} \quad i(t) &= C\dot{v}_o(t) \\
\text{Solving:} \quad v_i(t) &= RC\dot{v}_o(t) + v_o(t) \\
V_i(s) &= (1 + sRC)V_o(s) \\
H(s) &= \frac{V_o(s)}{V_i(s)} = \frac{1}{1 + sRC}
\end{aligned}
$$

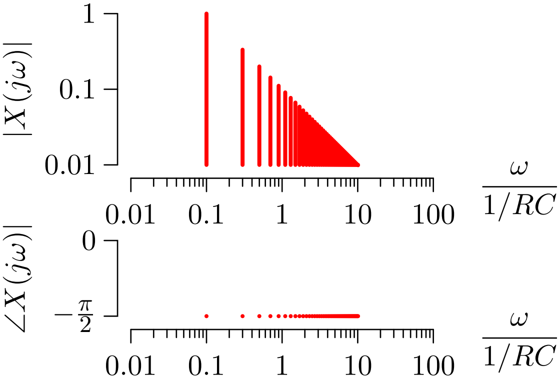


Lowpass Filtering

Let the input be a square wave.

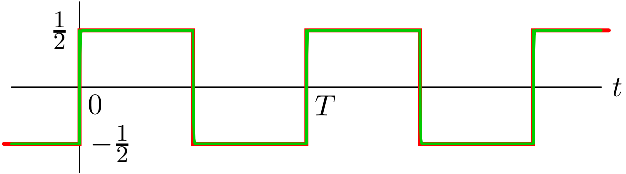


$$x(t) = \sum_{k \text{ odd}} \frac{1}{j\pi k} e^{j\omega_0 kt} \; ; \qquad \omega_0 = \frac{2\pi}{T}$$

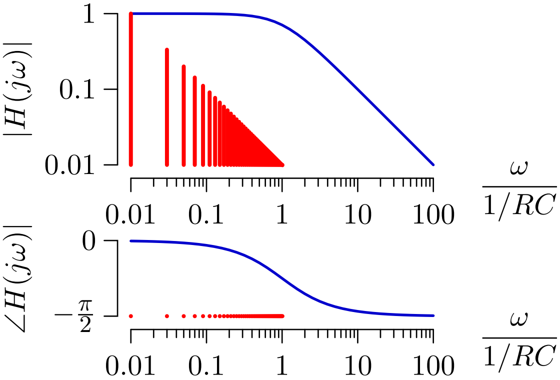


Lowpass Filtering

Low frequency square wave: $\omega_0 << 1/RC$.

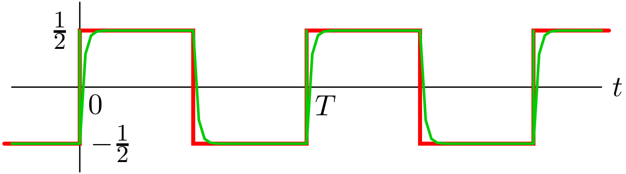


$$x(t) = \sum_{k \text{ odd}} \frac{1}{j\pi k} e^{j\omega_0 kt} \quad ; \qquad \omega_0 = \frac{2\pi}{T}$$

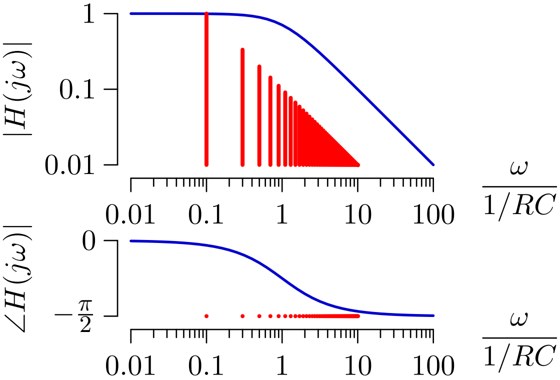


Lowpass Filtering

Higher frequency square wave: $\omega_0 < 1/RC$.

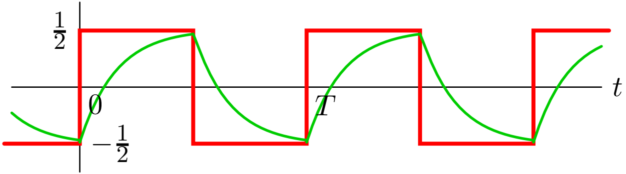


$$x(t) = \sum_{k \text{ odd}} \frac{1}{j\pi k} e^{j\omega_0 k t} \; ; \qquad \omega_0 = \frac{2\pi}{T}$$

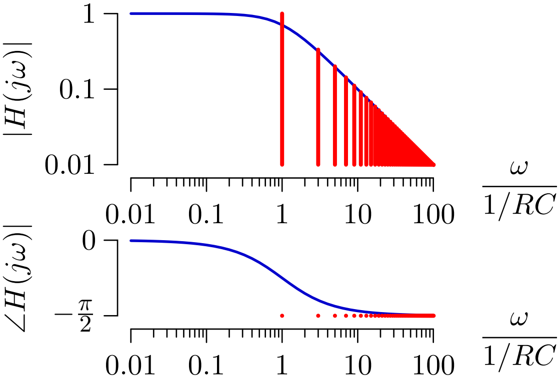


Lowpass Filtering

Still higher frequency square wave: $\omega_0 = 1/RC$.

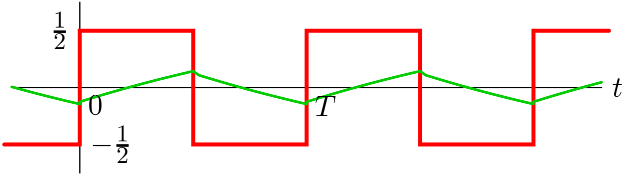


$$x(t) = \sum_{k \text{ odd}} \frac{1}{j\pi k} e^{j\omega_0 kt} \quad ; \qquad \omega_0 = \frac{2\pi}{T}$$

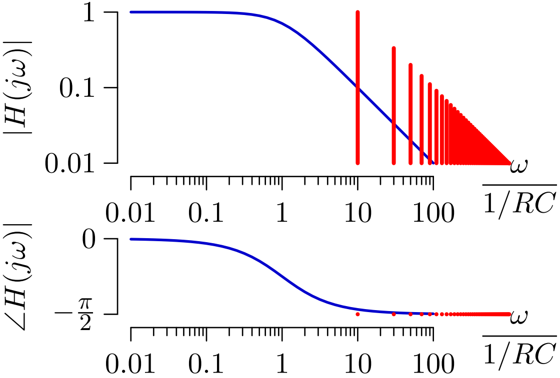


Lowpass Filtering

High frequency square wave: $\omega_0 > 1/RC$.

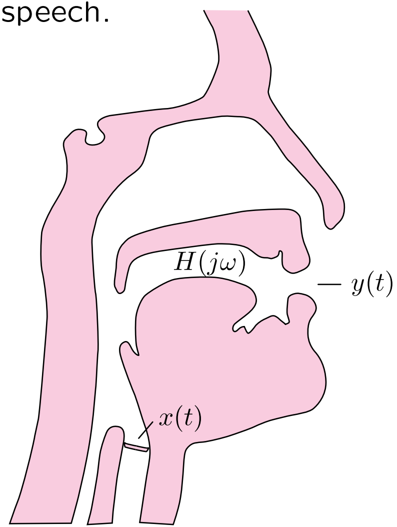


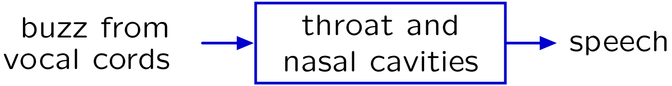
$$x(t) = \sum_{k \text{ odd}} \frac{1}{j\pi k} e^{j\omega_0 k t} \quad ; \qquad \omega_0 = \frac{2\pi}{T}$$



Source-Filter Model of Speech Production

Vibrations of the vocal cords are "filtered" by the mouth and nasal cavities to generate speech.





Filtering

LTI systems "filter" signals based on their frequency content.

Fourier transforms represent signals as sums of complex exponentials.

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

Complex exponentials are eigenfunctions of LTI systems.

$$e^{j\omega t} \to H(j\omega)e^{j\omega t}$$

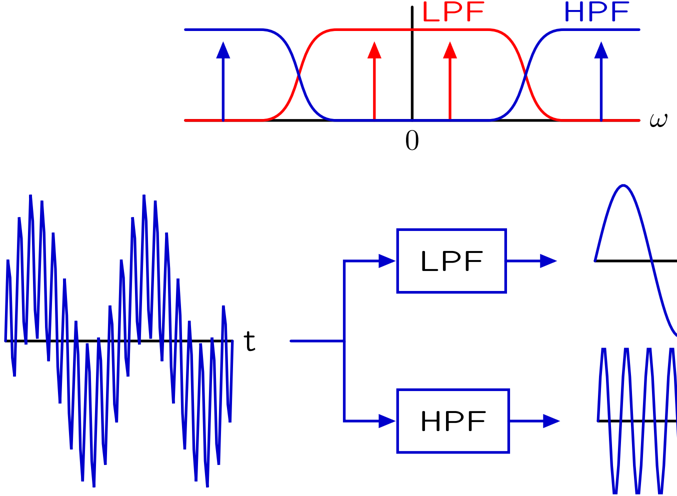
LTI systems "filter" signals by adjusting the amplitudes and phases of each frequency component.

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \;\; \to \;\; y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(j\omega)X(j\omega)e^{j\omega t}d\omega$$

Filtering

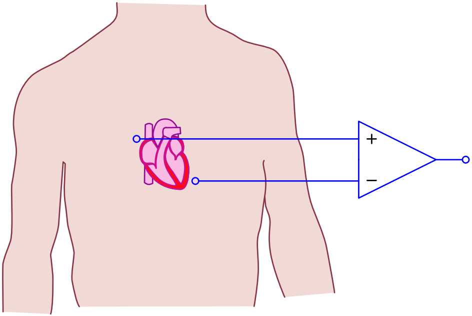
Systems can be designed to selectively pass certain frequency bands.

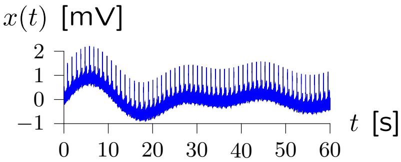
Examples: low-pass filter (LPF) and high-pass filter (HPF).



Filtering Example: Electrocardiogram

An electrocardiogram is a record of electrical potentials that are generated by the heart and measured on the surface of the chest.



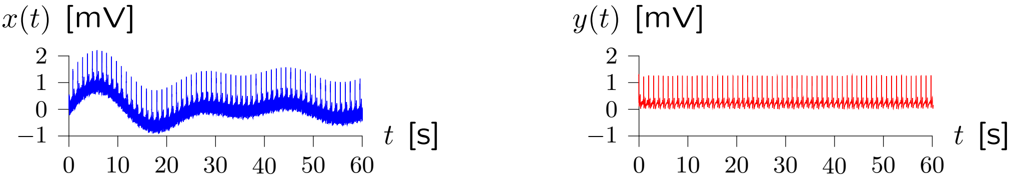


ECG and analysis by T. F. Weiss

Filtering Example: Electrocardiogram

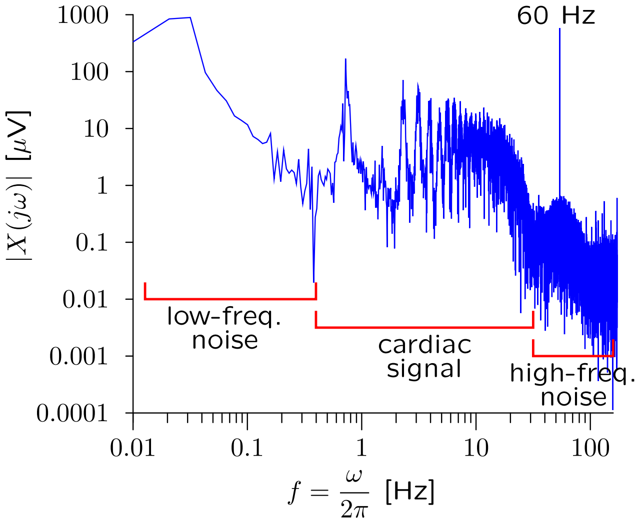
In addition to picking up electrical responses of the heart, electrodes on the skin also pick up a variety of other electrical signals that we regard as "noise."

We wish to design a filter to eliminate the noise.



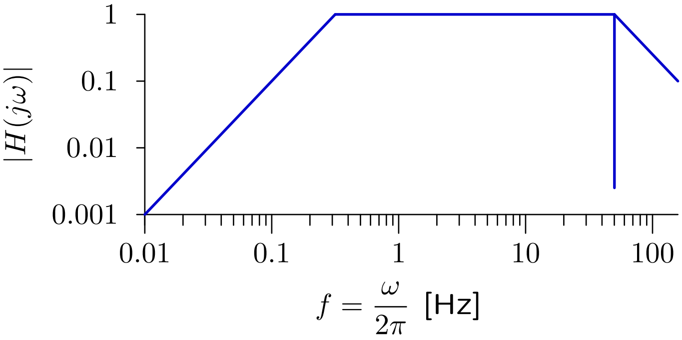
Filtering Example: Electrocardiogram

We can identify the “noise” by breaking the electrocardiogram into frequency components using the Fourier transform.



Filtering Example: Electrocardiogram

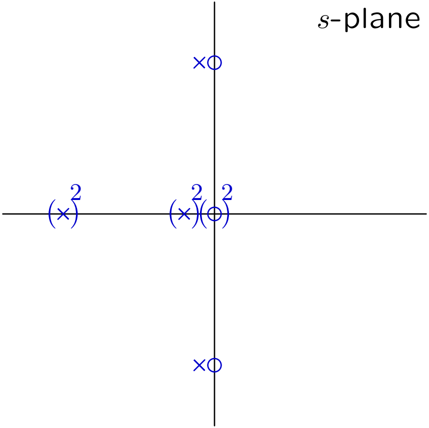
Filter design: low-pass filter + high-pass filter + notch.



Electrocardiogram: Check Yourself

Which poles and zeros are associated with

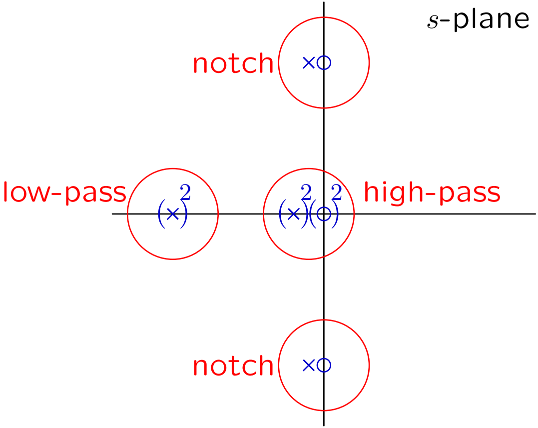
- the high-pass filter?
- the low-pass filter?
- the notch filter?



Electrocardiogram: Check Yourself

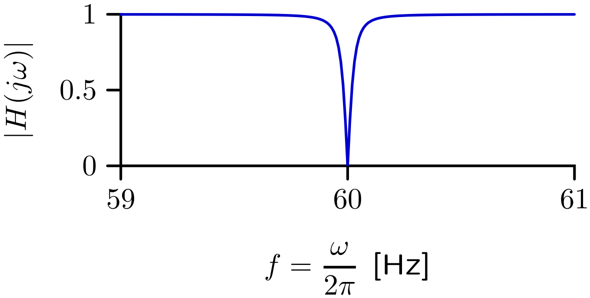
Which poles and zeros are associated with

- the high-pass filter?
- the low-pass filter?
- the notch filter?



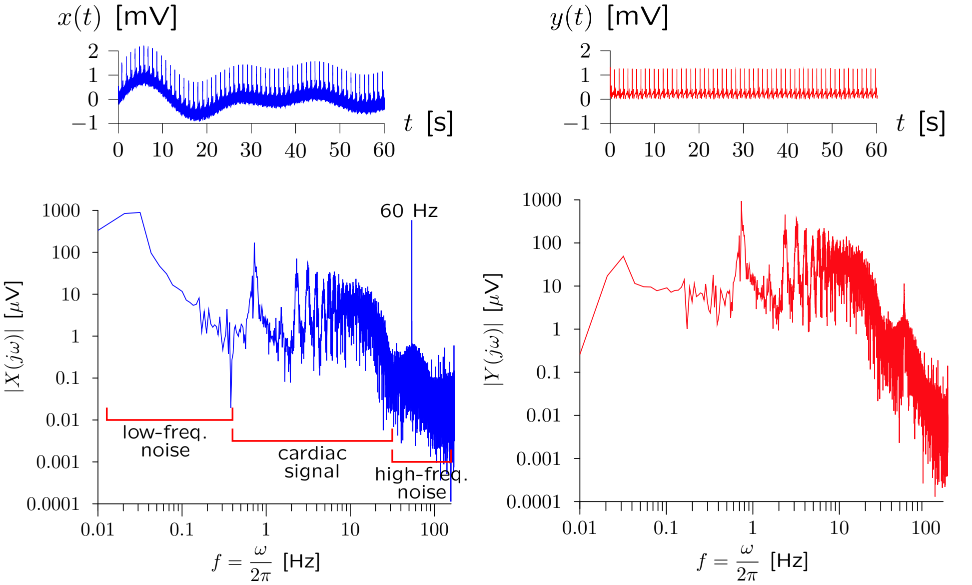
Filtering Example: Electrocardiogram

By placing the poles of the notch filter very close to the zeros, the width of the notch can be made quite small.



Filtering Example: Electrocardiogram

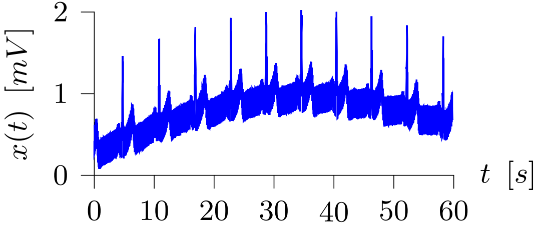
Comparision of filtered and unfiltered electrocardiograms.



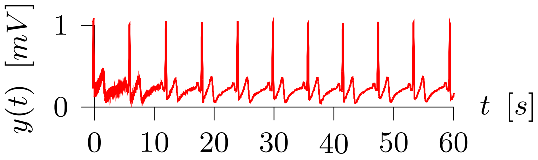
Filtering Example: Electrocardiogram

Reducing the frequency components that are not generated by the heart simplifies the output, making it easier to diagnose cardiac problems.

Unfiltered ECG



Filtered ECG



Continuous-Time Fourier Transform: Summary

Fourier transforms represent signals by their frequency content.

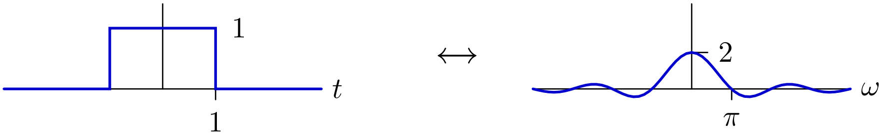
$\rightarrow$ useful for many signals, e.g., electrocardiogram.

$\rightarrow$ motivates representing a system as a filter.

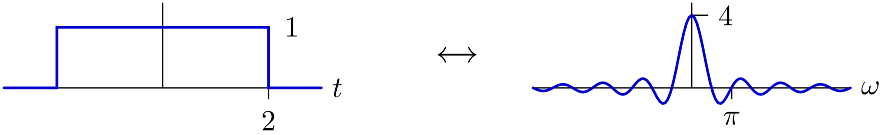
  $\rightarrow$ useful for many systems.

Visualizing the Fourier Transform

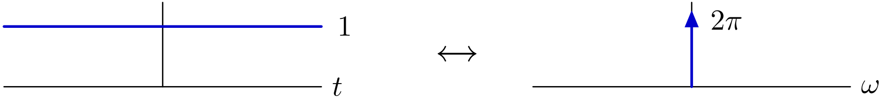
Fourier transforms provide alternate **views** of signals.



Pulses contain all frequencies except harmonics of 2π/width.



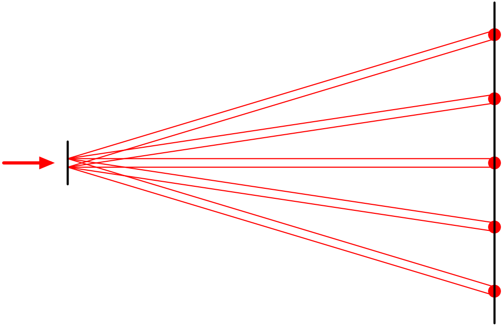
Wider pulses contain more low frequencies than narrow pulses.



Constants (in time) contain only frequencies at $\omega = 0$.

Fourier Transforms in Physics: Diffraction

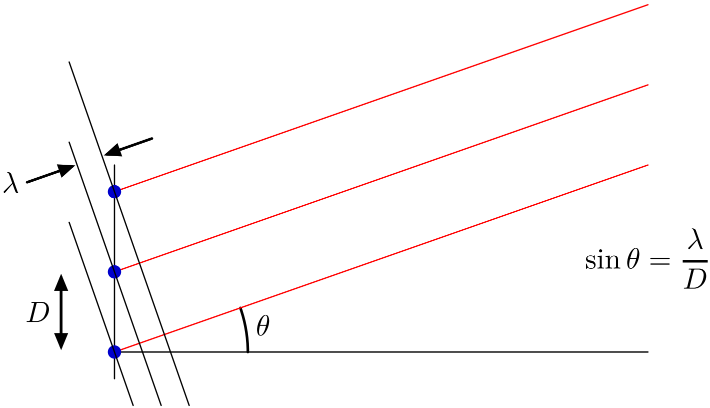
A diffraction grating breaks a laser beam input into multiple beams.



Demonstration.

Fourier Transforms in Physics: Diffraction

The grating has a periodic structure (period $= D$).



The “far field” image is formed by interference of scattered light.

Viewed from angle θ, the scatterers are separated by $D\sin\theta$.

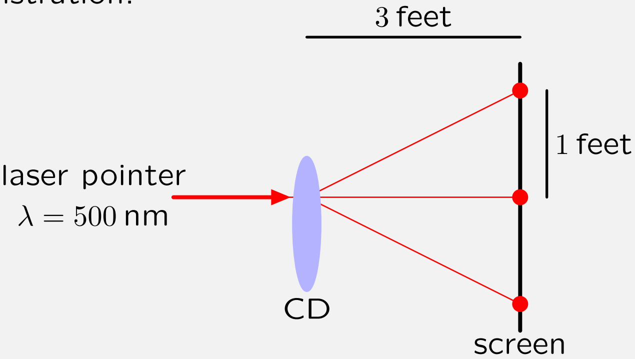
If this distance is an integer number of wavelengths $\lambda \rightarrow$ constructive interference.

Fourier Transforms in Physics: Diffraction

CD demonstration.

Check Yourself

CD demonstration.



What is the spacing of the tracks on the CD?

1. 160 nm
2. 1600 nm
3. 16μm
4. 160μm

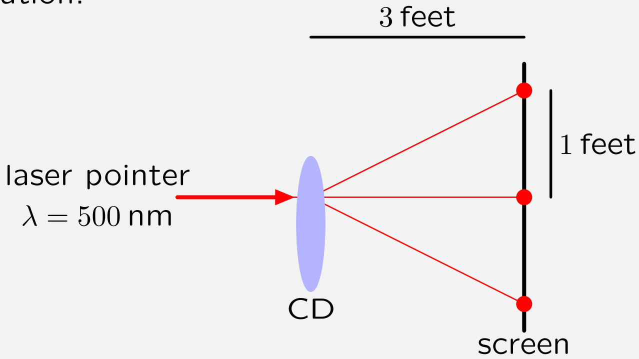
Check Yourself

What is the spacing of the tracks on the CD?

grating	$\tan\theta$	θ	$\sin\theta$	$D = \dfrac{500\,\text{nm}}{\sin\theta}$	manufacturing spec.
CD	$\frac{1}{3}$	0.32	0.31	$1613\,\text{nm}$	$1600\,\text{nm}$

Check Yourself

Demonstration.



What is the spacing of the tracks on the CD? 2.

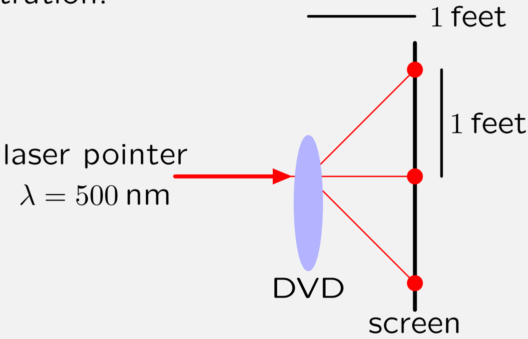
1. 160 nm
2. 1600 nm
3. 16μm
4. 160μm

Fourier Transforms in Physics: Diffraction

DVD demonstration.

Check Yourself

DVD demonstration.



What is track spacing on DVD divided by that for CD?

1. $4\times$
2. $2\times$
3. $\frac{1}{2}\times$
4. $\frac{1}{4}\times$

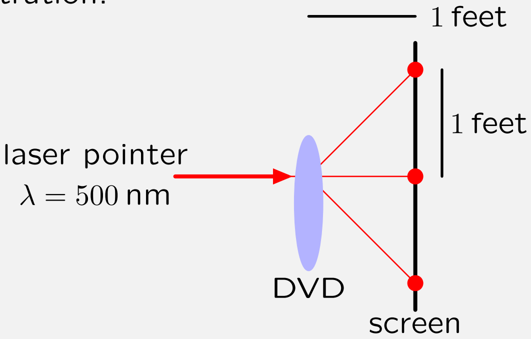
Check Yourself

What is spacing of tracks on DVD divided by that for CD?

grating	$\tan\theta$	θ	$\sin\theta$	$D = \dfrac{500\,\text{nm}}{\sin\theta}$	manufacturing spec.
CD	$\frac{1}{3}$	0.32	0.31	$1613\,\text{nm}$	$1600\,\text{nm}$
DVD	1	0.78	0.71	$704\,\text{nm}$	$740\,\text{nm}$

Check Yourself

DVD demonstration.

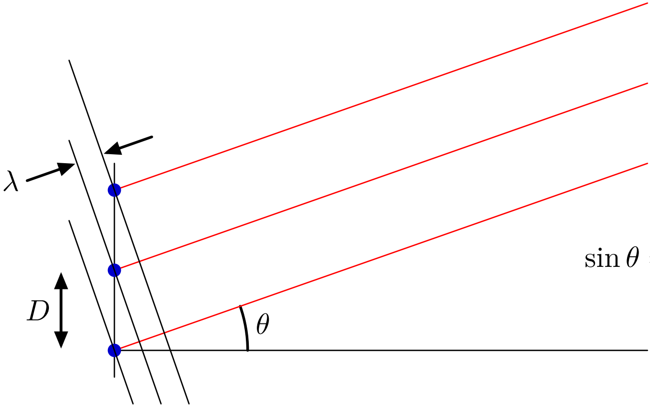


What is track spacing on DVD divided by that for CD? 3

1. $4\times$
2. $2\times$
3. $\frac{1}{2}\times$
4. $\frac{1}{4}\times$

Fourier Transforms in Physics: Diffraction

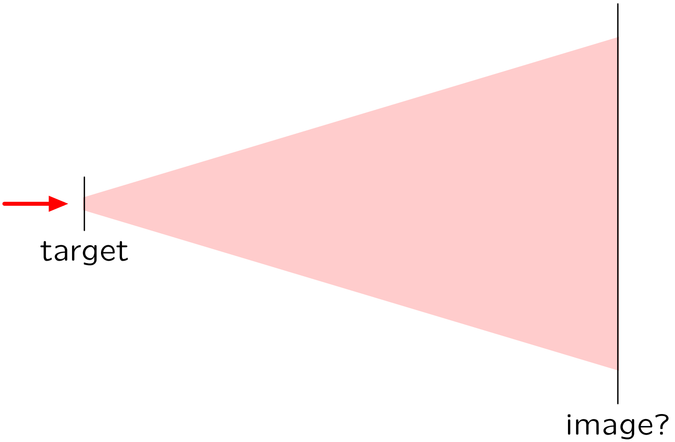
Macroscopic information in the far field provides microscopic (invisible) information about the grating.



$$\sin\theta = \frac{\lambda}{D}$$

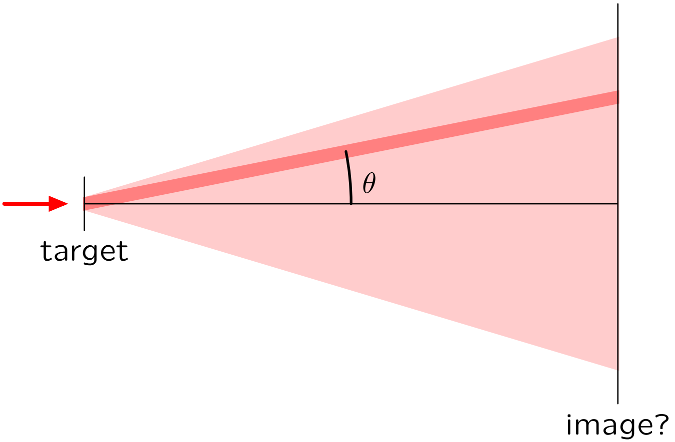
Fourier Transforms in Physics: Crystallography

What if the target is more complicated than a grating?



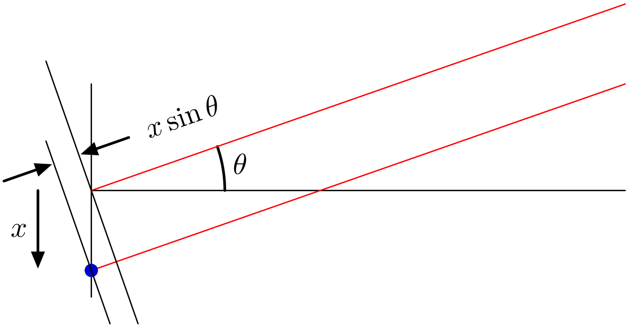
Fourier Transforms in Physics: Crystallography

Part of image at angle θ has contributions for all parts of the target.



Fourier Transforms in Physics: Crystallography

The phase of light scattered from different parts of the target undergo different amounts of phase delay.



Phase at a point x is delayed (i.e., negative) relative to that at 0:

$$\phi = -2\pi \frac{x \sin\theta}{\lambda}$$

Fourier Transforms in Physics: Crystallography

Total light $F(\theta)$ at angle θ is the integral of amount scattered from each part of the target ($f(x)$) appropriately shifted in phase.

$$F(\theta) = \int f(x) e^{-j2\pi \frac{x \sin\theta}{\lambda}} dx$$

Assume small angles so $\sin\theta \approx \theta$.

Let $\omega = 2\pi \frac{\theta}{\lambda}$.

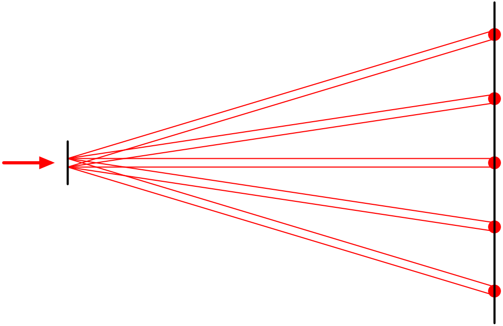
Then the pattern of light at the detector is

$$F(\omega) = \int f(x) e^{-j\omega x} dx$$

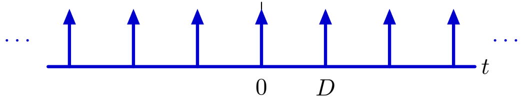
which is the Fourier transform of $f(x)$!

Fourier Transforms in Physics: Diffraction

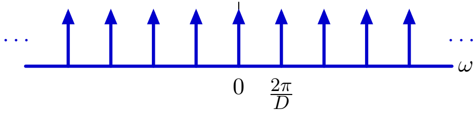
There is a Fourier transform relation between this structure and the far-field intensity pattern.



grating $\approx$ impulse train with pitch D



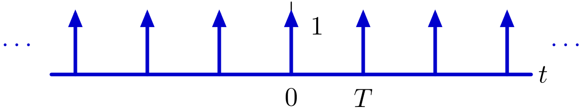
far-field intensity $\approx$ impulse train with reciprocal pitch $\propto \frac{\lambda}{D}$



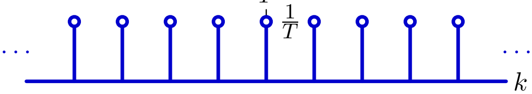
Impulse Train

The Fourier transform of an impulse train is an impulse train.

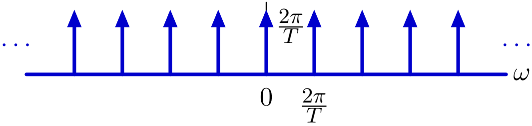
$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$



$$a_k = \frac{1}{T} \quad \forall \; k$$



$$X(j\omega) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{T} \delta(\omega - k\frac{2\pi}{T})$$

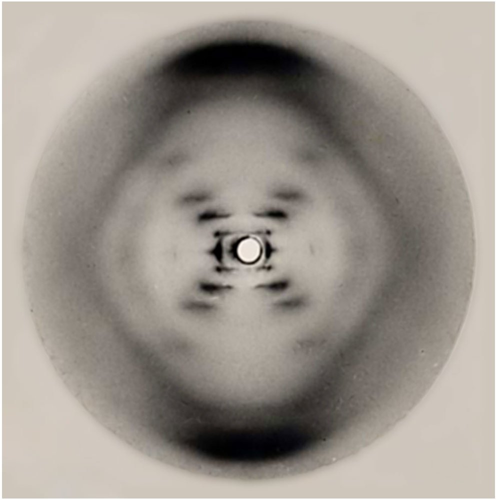


Two Dimensions

Demonstration: 2D grating.

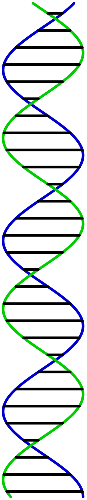
An Historic Fourier Transform

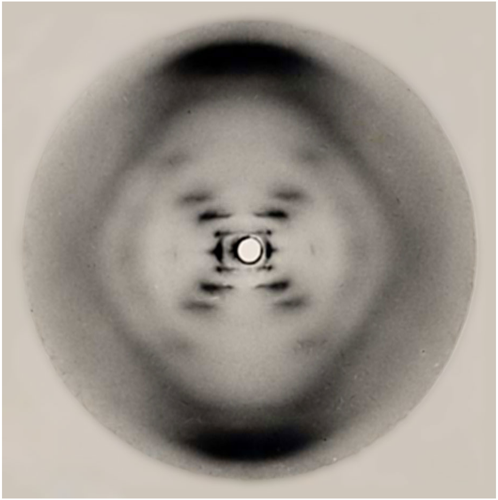
Taken by Rosalind Franklin, this image sparked Watson and Crick's insight into the double helix.



An Historic Fourier Transform

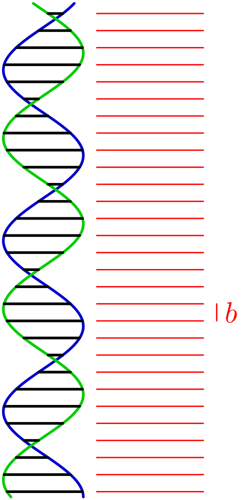
This is an x-ray crystallographic image of DNA, and it shows the Fourier transform of the structure of DNA.

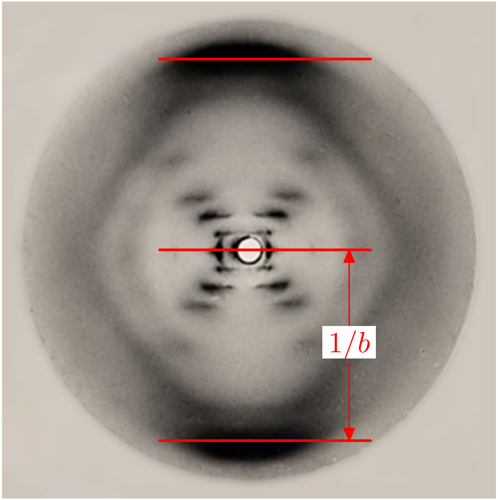




An Historic Fourier Transform

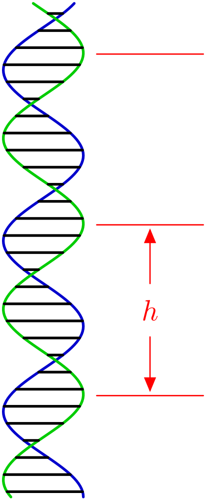
High-frequency bands indicate repreating structure of base pairs.

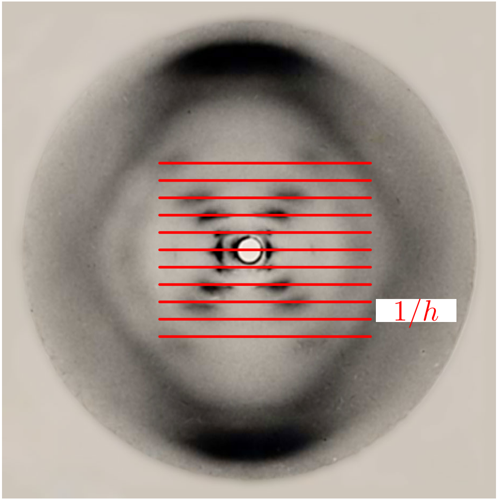




An Historic Fourier Transform

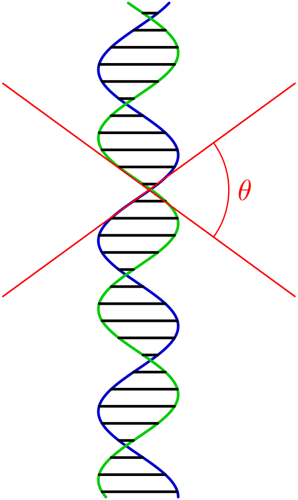
Low-frequency bands indicate a lower frequency repeating structure.

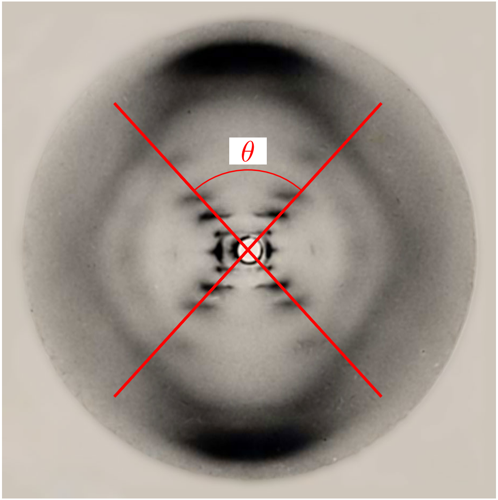




An Historic Fourier Transform

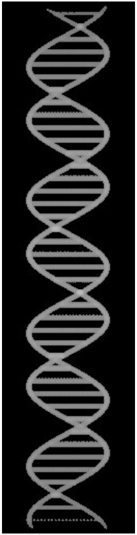
Tilt of low-frequency bands indicates tilt of low-frequency repeating structure: the double helix!

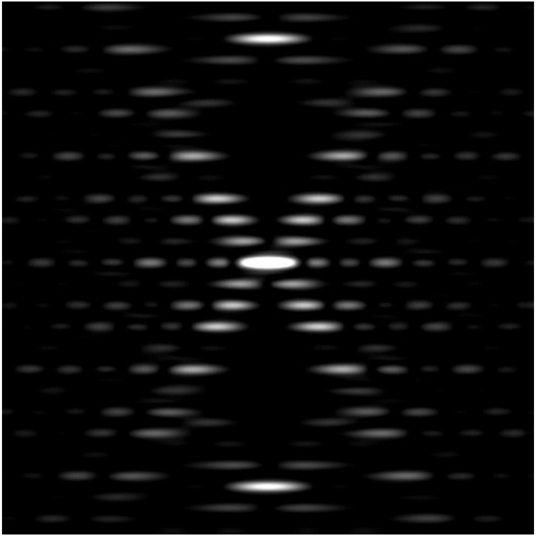




Simulation

Easy to calculate relation between structure and Fourier transform.





Fourier Transform Summary

Represent signals by their frequency content.

Key to “filtering,” and to signal-processing in general.

Important in many physical phenomenon: x-ray crystallography.