6.003: Signals and Systems

DT Fourier Representations

April 15, 2010

Mid-term Examination #3

Wednesday, April 28, 7:30-9:30pm, 34-101.

No recitations on the day of the exam.

Coverage: Lectures 1–20
Recitations 1–20
Homeworks 1–11

Homework 11 will not collected or graded. Solutions will be posted.

Closed book: 3 pages of notes ($8\frac{1}{2} \times 11$ inches; front and back).

Designed as 1-hour exam; two hours to complete.

Review sessions during open office hours.

Conflict? Contact freeman@mit.edu before Friday, April 23, 5pm.

Review: DT Frequency Response

The frequency response of a DT LTI system is the value of the system function evaluated on the unit circle.

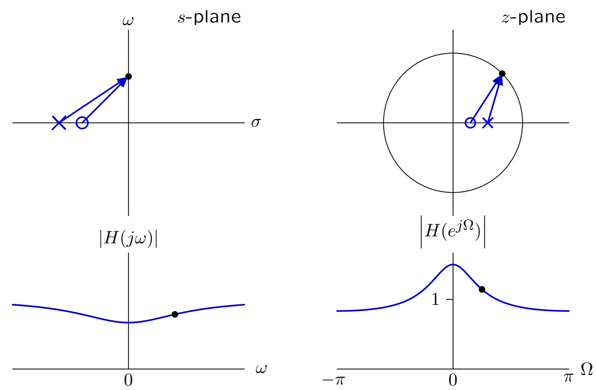
$$\cos(\Omega n) \rightarrow \boxed{H(z)} \rightarrow |H(e^{j\Omega})| \cos\left(\Omega n + \angle H(e^{j\Omega})\right)$$

$$H(e^{j\Omega}) = H(z)|_{z=e^{j\Omega}}$$

Comparision of CT and DT Frequency Responses

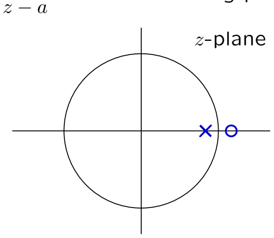
CT frequency response: $H(s)$ on the imaginary axis, i.e., $s = j\omega$.

DT frequency response: $H(z)$ on the unit circle, i.e., $z = e^{j\Omega}$.



Check Yourself

A system $H(z) = \dfrac{1-az}{z-a}$ has the following pole-zero diagram.

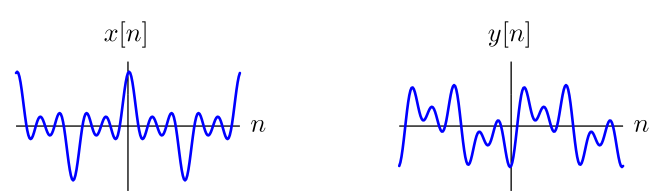


Classify this system as one of the following filter types.

1. high pass
2. low pass
3. band pass
4. all pass
5. band stop
0. none of the above

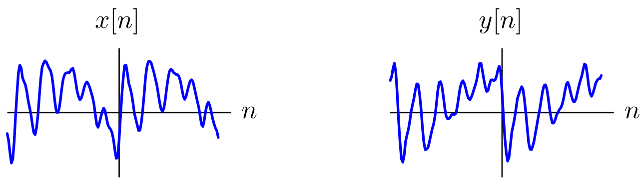
Effects of Phase

$$x[n] \rightarrow \boxed{H(z) = \frac{1-az}{z-a}} \rightarrow y[n]$$



Effects of Phase

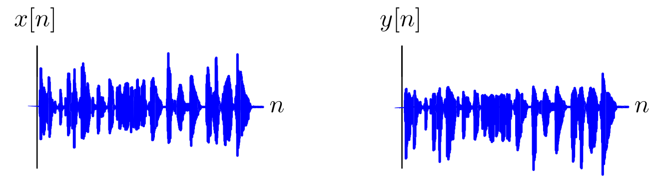
$$x[n] \longrightarrow \boxed{H(z) = \frac{1-az}{z-a}} \longrightarrow y[n]$$



http://public.research.att.com/~ttsweb/tts/demo.php

Effects of Phase

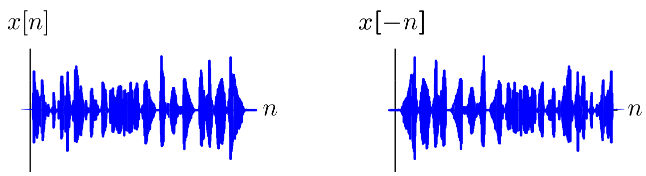
$$x[n] \longrightarrow \boxed{H(z) = \frac{1-az}{z-a}} \longrightarrow y[n]$$



artificial speech synthesized by Robert Donovan

Effects of Phase

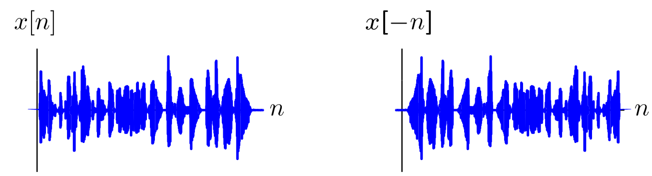
$$x[n] \longrightarrow \boxed{???} \longrightarrow y[n] = x[-n]$$



artificial speech synthesized by Robert Donovan

Effects of Phase

$$x[n] \longrightarrow \boxed{???} \longrightarrow y[n] = x[-n]$$



How are the phases of X and Y related?

Review: Periodicity

DT frequency responses are periodic functions of Ω, with period 2π.

If $\Omega_2 = \Omega_1 + 2\pi k$ where k is an integer then

$$H(e^{j\Omega_2}) = H(e^{j(\Omega_1+2\pi k)}) = H(e^{j\Omega_1}e^{j2\pi k}) = H(e^{j\Omega_1})$$

The periodicity of $H(e^{j\Omega})$ results because $H(e^{j\Omega})$ is a function of $e^{j\Omega}$, which is itself periodic in Ω. Thus DT complex exponentials have many "aliases."

$$e^{j\Omega_2} = e^{j(\Omega_1+2\pi k)} = e^{j\Omega_1}e^{j2\pi k} = e^{j\Omega_1}$$

Because of this aliasing, there is a "highest" DT frequency: $\Omega = \pi$.

Review: Periodic Sinusoids

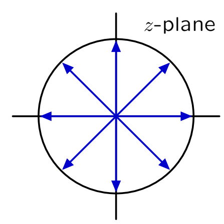
There are N distinct DT complex exponentials with period N.

If $e^{j\Omega n}$ is periodic in N then

$$e^{j\Omega n} = e^{j\Omega(n+N)} = e^{j\Omega n}e^{j\Omega N}$$

and $e^{j\Omega N}$ must be 1, and Ω must be one of the N^{th} roots of 1.

Example: $N = 8$



Review: DT Fourier Series

DT Fourier series represent DT signals in terms of the amplitudes and phases of harmonic components.

DT Fourier Series

$$a_k = a_{k+N} = \frac{1}{N}\sum_{n=<N>} x[n]e^{-j\Omega_0 n} \;;\; \Omega_0 = \frac{2\pi}{N} \qquad \text{("analysis" equation)}$$

$$x[n] = x[n+N] = \sum_{k=<N>} a_k e^{jk\Omega_0 n} \qquad \text{("synthesis" equation)}$$

DT Fourier Series

DT Fourier series have simple matrix interpretations.

$$x[n] = x[n+4] = \sum_{k=<4>} a_k e^{jk\Omega_0 n} = \sum_{k=<4>} a_k e^{jk\frac{2\pi}{4}n} = \sum_{k=<4>} a_k j^{kn}$$

$$\begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$a_k = a_{k+4} = \frac{1}{4}\sum_{n=<4>} x[n]e^{-jk\Omega_0 n} = \frac{1}{4}\sum_{n=<4>} e^{-jk\frac{2\pi}{N}n} = \frac{1}{4}\sum_{n=<4>} x[n]j^{-kn}$$

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix}$$

These matrices are inverses of each other.

Scaling

DT Fourier series are important computational tools.
However, the DT Fourier series do not scale well with the length N.

$$a_k = a_{k+2} = \frac{1}{2}\sum_{n=<2>} x[n]e^{-jk\Omega_0 n} = \frac{1}{2}\sum_{n=<2>} e^{-jk\frac{2\pi}{2}n} = \frac{1}{2}\sum_{n=<2>} x[n](-1)^{-kn}$$

$$\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \end{bmatrix}$$

$$a_k = a_{k+4} = \frac{1}{4}\sum_{n=<4>} x[n]e^{-jk\Omega_0 n} = \frac{1}{4}\sum_{n=<4>} e^{-jk\frac{2\pi}{4}n} = \frac{1}{4}\sum_{n=<4>} x[n]j^{-kn}$$

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix}$$

Number of multiples increases as N^2.

Fast Fourier "Transform"

Exploit structure of Fourier series to simplify its calculation.

Divide FS of length $2N$ into two of length N (divide and conquer).

Matrix formulation of 8-point FS:

$$\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \end{bmatrix} = \begin{bmatrix} W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 \\ W_8^0 & W_8^1 & W_8^2 & W_8^3 & W_8^4 & W_8^5 & W_8^6 & W_8^7 \\ W_8^0 & W_8^2 & W_8^4 & W_8^6 & W_8^0 & W_8^2 & W_8^4 & W_8^6 \\ W_8^0 & W_8^3 & W_8^6 & W_8^1 & W_8^4 & W_8^7 & W_8^2 & W_8^5 \\ W_8^0 & W_8^4 & W_8^0 & W_8^4 & W_8^0 & W_8^4 & W_8^0 & W_8^4 \\ W_8^0 & W_8^5 & W_8^2 & W_8^7 & W_8^4 & W_8^1 & W_8^6 & W_8^3 \\ W_8^0 & W_8^6 & W_8^4 & W_8^2 & W_8^0 & W_8^6 & W_8^4 & W_8^2 \\ W_8^0 & W_8^7 & W_8^6 & W_8^5 & W_8^4 & W_8^3 & W_8^2 & W_8^1 \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \\ x[4] \\ x[5] \\ x[6] \\ x[7] \end{bmatrix}$$

where $W_N = e^{-j\frac{2\pi}{N}}$

$8 \times 8 = 64$ multiplications

FFT

Divide into two 4-point series (divide and conquer).

Even-numbered entries in $x[n]$:

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^0 & W_4^2 \\ W_4^0 & W_4^3 & W_4^2 & W_4^1 \end{bmatrix} \begin{bmatrix} x[0] \\ x[2] \\ x[4] \\ x[6] \end{bmatrix}$$

Odd-numbered entries in $x[n]$:

$$\begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^0 & W_4^2 \\ W_4^0 & W_4^3 & W_4^2 & W_4^1 \end{bmatrix} \begin{bmatrix} x[1] \\ x[3] \\ x[5] \\ x[7] \end{bmatrix}$$

Sum of multiplications $= 2 \times (4 \times 4) = 32$: fewer than the previous 64.

FFT

Break the original 8-point DTFS coefficients c_k into two parts:

$$c_k = d_k + e_k$$

where d_k comes from the even-numbered $x[n]$ (e.g., a_k) and e_k comes from the odd-numbered $x[n]$ (e.g., b_k)

FFT

The 4-point DTFS coefficients a_k of the even-numbered $x[n]$

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^0 & W_4^2 \\ W_4^0 & W_4^3 & W_4^2 & W_4^1 \end{bmatrix} \begin{bmatrix} x[0] \\ x[2] \\ x[4] \\ x[6] \end{bmatrix} = \begin{bmatrix} W_8^0 & W_8^0 & W_8^0 & W_8^0 \\ W_8^0 & W_8^2 & W_8^4 & W_8^6 \\ W_8^0 & W_8^4 & W_8^0 & W_8^4 \\ W_8^0 & W_8^6 & W_8^4 & W_8^2 \end{bmatrix} \begin{bmatrix} x[0] \\ x[2] \\ x[4] \\ x[6] \end{bmatrix}$$

contribute to the 8-point DTFS coefficients d_k:

$$\boxed{\begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \\ d_7 \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}} = \begin{bmatrix} W_8^0 & W_8^0 & W_8^0 & W_8^0 \\ W_8^0 & W_8^2 & W_8^4 & W_8^6 \\ W_8^0 & W_8^4 & W_8^0 & W_8^4 \\ W_8^0 & W_8^6 & W_8^4 & W_8^2 \\ W_8^0 & W_8^0 & W_8^0 & W_8^0 \\ W_8^0 & W_8^2 & W_8^4 & W_8^6 \\ W_8^0 & W_8^4 & W_8^0 & W_8^4 \\ W_8^0 & W_8^6 & W_8^4 & W_8^2 \end{bmatrix} \begin{bmatrix} x[0] \\ x[2] \\ x[4] \\ x[6] \end{bmatrix}$$

FFT

The e_k components result from the odd-number entries in $x[n]$.

$$\begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^0 & W_4^2 \\ W_4^0 & W_4^3 & W_4^2 & W_4^1 \end{bmatrix} \begin{bmatrix} x[1] \\ x[3] \\ x[5] \\ x[7] \end{bmatrix} = \begin{bmatrix} W_8^0 & W_8^0 & W_8^0 & W_8^0 \\ W_8^0 & W_8^2 & W_8^4 & W_8^6 \\ W_8^0 & W_8^4 & W_8^0 & W_8^4 \\ W_8^0 & W_8^6 & W_8^4 & W_8^2 \end{bmatrix} \begin{bmatrix} x[1] \\ x[3] \\ x[5] \\ x[7] \end{bmatrix}$$

$$\boxed{\begin{bmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ e_7 \end{bmatrix} = \begin{bmatrix} W_8^0\, b_0 \\ W_8^1\, b_1 \\ W_8^2\, b_2 \\ W_8^3\, b_3 \\ W_8^4\, b_0 \\ W_8^5\, b_1 \\ W_8^6\, b_2 \\ W_8^7\, b_3 \end{bmatrix}} = \begin{bmatrix} W_8^0 & W_8^0 & W_8^0 & W_8^0 \\ W_8^1 & W_8^3 & W_8^5 & W_8^7 \\ W_8^2 & W_8^6 & W_8^2 & W_8^6 \\ W_8^3 & W_8^1 & W_8^7 & W_8^5 \\ W_8^4 & W_8^4 & W_8^4 & W_8^4 \\ W_8^5 & W_8^7 & W_8^1 & W_8^3 \\ W_8^6 & W_8^2 & W_8^6 & W_8^2 \\ W_8^7 & W_8^5 & W_8^3 & W_8^1 \end{bmatrix} \begin{bmatrix} x[1] \\ x[3] \\ x[5] \\ x[7] \end{bmatrix}$$

FFT

Combine a_k and b_k to get c_k.

$$\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \end{bmatrix} = \begin{bmatrix} d_0 + e_0 \\ d_1 + e_1 \\ d_2 + e_2 \\ d_3 + e_3 \\ d_4 + e_4 \\ d_5 + e_5 \\ d_6 + e_6 \\ d_7 + e_7 \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} W_8^0\, b_0 \\ W_8^1\, b_1 \\ W_8^2\, b_2 \\ W_8^3\, b_3 \\ W_8^4\, b_0 \\ W_8^5\, b_1 \\ W_8^6\, b_2 \\ W_8^7\, b_3 \end{bmatrix}$$

FFT procedure:

- compute a_k and b_k: $2 \times (4 \times 4) = 32$ multiplies
- combine $c_k = a_k + W_8^k\, b_k$: 8 multiples
- total 40 multiplies: fewer than the orginal $8 \times 8 = 64$ multiplies

Scaling of FFT algorithm

How does the new algorithm scale?

Let $M(N) =$ number of multiples to perform an N point FFT.

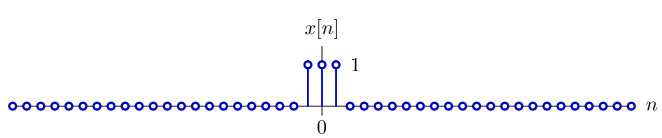
$$\begin{aligned} M(1) &= 0 \\ M(2) &= 2M(1) + 2 = 2 \\ M(4) &= 2M(2) + 4 = 2 \times 4 \\ M(8) &= 2M(4) + 8 = 3 \times 8 \\ M(16) &= 2M(8) + 16 = 4 \times 16 \\ M(32) &= 2M(16) + 32 = 5 \times 32 \\ M(64) &= 2M(32) + 64 = 6 \times 64 \\ M(128) &= 2M(64) + 128 = 7 \times 128 \\ &\ \ \cdots \\ M(N) &= (\log_2 N) \times N \end{aligned}$$

Significantly smaller than N^2 for N large.

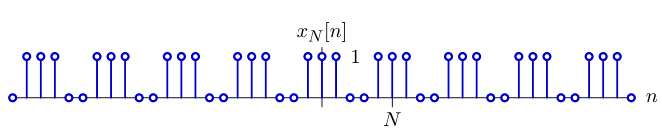
Fourier Transform: Generalize to Aperiodic Signals

An aperiodic signal can be thought of as periodic with infinite period.

Let $x[n]$ represent an aperiodic signal DT signal.



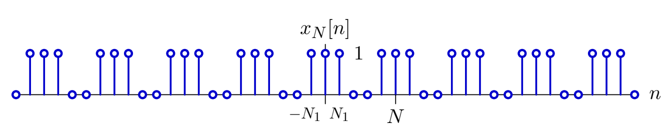
"Periodic extension": $x_N[n] = \sum_{k=-\infty}^{\infty} x[n + kN]$



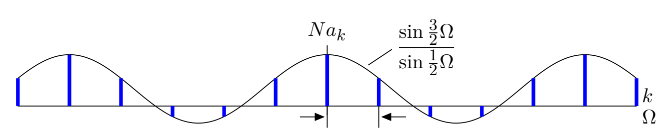
Then $x[n] = \lim_{N \to \infty} x_N[n]$.

Fourier Transform

Represent $x_N[n]$ by its Fourier series.



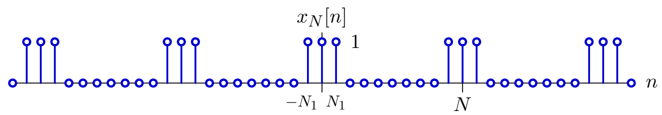
$$a_k = \frac{1}{N} \sum_{N} x_N[n] e^{-j\frac{2\pi}{N}kn} = \frac{1}{N} \sum_{n=-N_1}^{N_1} e^{-j\frac{2\pi}{N}kn} = \frac{1}{N} \frac{\sin\left(\left(N_1 + \frac{1}{2}\right)\Omega\right)}{\sin \frac{1}{2}\Omega}$$



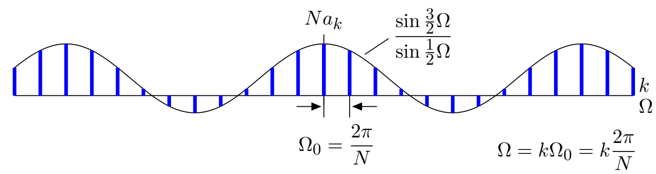
$\Omega_0 = \frac{2\pi}{N}$ $\qquad$ $\Omega = k\Omega_0 = k\frac{2\pi}{N}$

Fourier Transform

Doubling period doubles # of harmonics in given frequency interval.

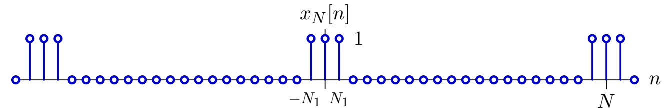


$$a_k = \frac{1}{N}\sum_{N} x_N[n]e^{-j\frac{2\pi}{N}kn} = \frac{1}{N}\sum_{n=-N_1}^{N_1} e^{-j\frac{2\pi}{N}kn} = \frac{1}{N}\frac{\sin\left(N_1+\frac{1}{2}\right)\Omega}{\sin\frac{1}{2}\Omega}$$

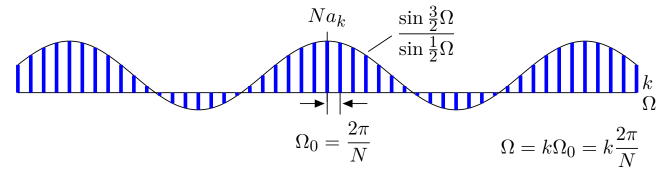


Fourier Transform

As $N \to \infty$, discrete harmonic amplitudes $\to$ a continuum $E(\Omega)$.



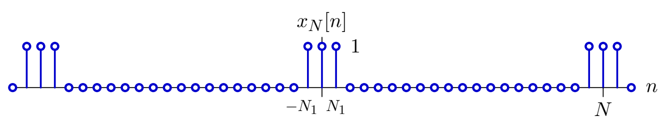
$$a_k = \frac{1}{N}\sum_{N} x_N[n]e^{-j\frac{2\pi}{N}kn} = \frac{1}{N}\sum_{n=-N_1}^{N_1} e^{-j\frac{2\pi}{N}kn} = \frac{1}{N}\frac{\sin\left(N_1+\frac{1}{2}\right)\Omega}{\sin\frac{1}{2}\Omega}$$

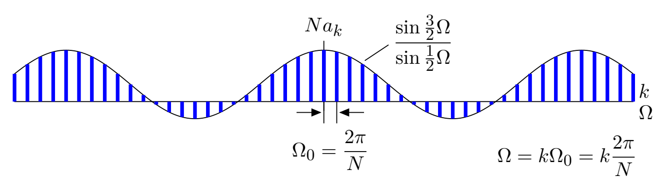


$$Na_k = \sum_{n=<N>} x[n]e^{-j\frac{2\pi}{N}kn} = \sum_{n=<N>} x[n]e^{-j\Omega n} = E(\Omega)$$

Fourier Transform

As $N \to \infty$, synthesis sum $\to$ integral.





$$Na_k = \sum_{n=<N>} x[n]e^{-j\frac{2\pi}{N}kn} = \sum_{n=<N>} x[n]e^{-j\Omega n} = E(\Omega)$$

$$x[n] = \sum_{k=<N>} \underbrace{\frac{1}{N}E(\Omega)}_{a_k} e^{j\frac{2\pi}{N}kn} = \sum_{k=<N>} \frac{\Omega_0}{2\pi}E(\Omega)e^{j\Omega n} \to \frac{1}{2\pi}\int_{2\pi} E(\Omega)e^{j\Omega n}d\Omega$$

Fourier Transform

Replacing $E(\Omega)$ by $X(e^{j\Omega})$ yields the DT Fourier transform relations.

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$$ ("analysis" equation)

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\Omega})e^{j\Omega n}d\Omega$$ ("synthesis" equation)

Relation between Fourier and Z Transforms

If the Z transform of a signal exists and if the ROC includes the unit circle, then the Fourier transform is equal to the Z transform evaluated on the unit circle.

Z transform:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

DT Fourier transform:

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} = H(z)\big|_{z=e^{j\Omega}}$$

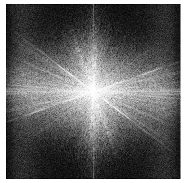
Relation between Fourier and Z Transforms

Fourier transform "inherits" properties of Z transform.

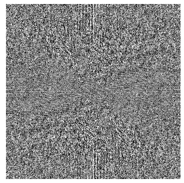
Property	$x[n]$	$X(z)$	$X(e^{j\Omega})$
Linearity	$ax_1[n]+bx_2[n]$	$aX_1(s)+bX_2(s)$	$aX_1(e^{j\Omega})+bX_2(e^{j\Omega})$
Time shift	$x[n-n_0]$	$z^{-n_0}X(z)$	$e^{-j\Omega n_0}X(e^{j\Omega})$
Multiply by n	$nx[n]$	$-z\frac{d}{dz}X(z)$	$j\frac{d}{d\Omega}X(e^{j\Omega})$
Convolution	$(x_1 * x_2)[n]$	$X_1(z)\times X_2(z)$	$X_1(e^{j\Omega})\times X_2(e^{j\Omega})$

DT Fourier Series of Images

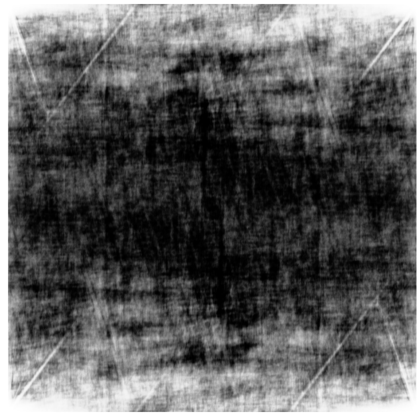
Magnitude



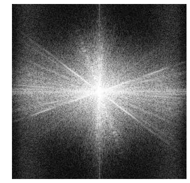
Angle



DT Fourier Series of Images



Magnitude

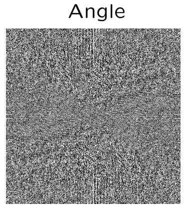


Uniform Angle

DT Fourier Series of Images

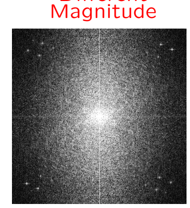
Uniform Magnitude

Angle

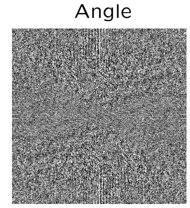


DT Fourier Series of Images

Different Magnitude



Angle



Fourier Representations: Summary

Thinking about signals by their frequency content and systems as filters has a large number of practical applications.