6.003 (Spring 2010)

Quiz #1 *March 3, 2010*

Name:

Kerberos Username:

Please circle your section number:

Section	*Instructor*	*Time*
1	Peter Hagelstein	10 am
2	Peter Hagelstein	11 am
3	Rahul Sarpeshkar	1 pm
4	Rahul Sarpeshkar	2 pm

Grades will be determined by the correctness of your answers (explanations are not required).

Partial credit will be given for ANSWERS that demonstrate some but not all of the important conceptual issues.

You have **two hours.**

Please put your initials on all subsequent sheets.

Enter your answers in the boxes.

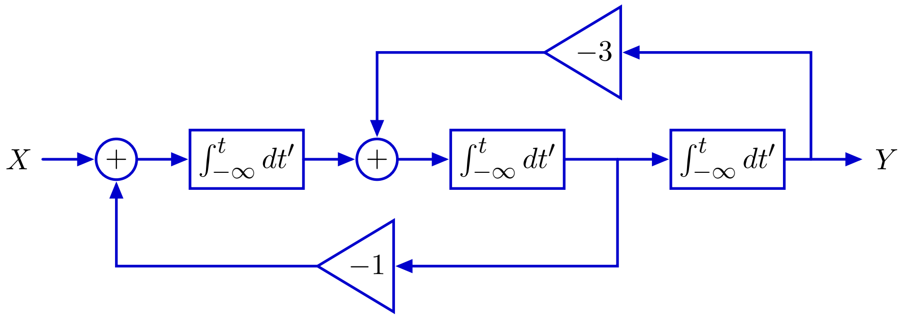
This quiz is closed book, but you may use one 8.5×11 sheet of paper (two sides).

No calculators, computers, cell phones, music players, or other aids.

1	/25
2	/25
3	/25
4	/25
Total	/100

1. Block diagram *[25 points]*

Consider the system represented by the following block diagram.



Part a. Is it possible to represent this system with a linear differential equation with constant coefficients?

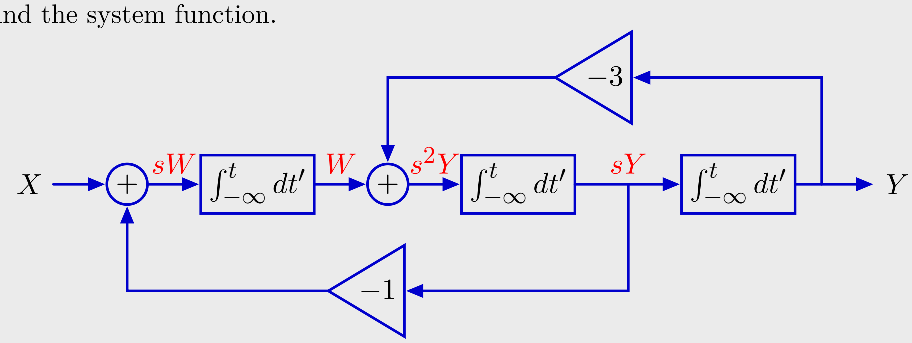
Yes or **No**: Yes

If yes, enter the differential equation in the box below.

$$\frac{d^3y(t)}{dt^3} + 4\frac{dy(t)}{dt} = x(t)$$

If no, briefly explain why not.

First find the system function.



$$sW = X - sY$$
$$s^2Y = W - 3Y$$

Multiply the second equation by s and substitute the first:

$$s^3Y = X - sY - 3sY = X - 4sY$$

$$\frac{d^3y(t)}{dt^3} + 4\frac{dy(t)}{dt} = x(t)$$

Part b. Determine the response $y(t)$ when the system starts at rest and the input $x(t) = \delta(t)$.

$y(t)$: $\frac{1}{4}\big(1 - \cos(2t)\big)\, u(t)$

$$s^3 Y = X - sY - 3sY = X - 4sY$$

Solve:

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s(s^2+4)}$$

Expand using partial fractions:

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s(s^2+4)} = \frac{1/4}{s} - \frac{1/8}{s+j2} - \frac{1/8}{s-j2}$$

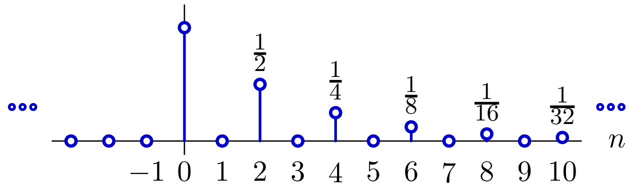
Invert each term:

$$h(t) = \frac{1}{4}u(t) - \frac{1}{8}e^{-jt}u(t) - \frac{1}{8}e^{jt}u(t) = \frac{1}{4}\big(1 - \cos(2t)\big)\, u(t)$$

2. Unit-sample response *[25 points]*

Consider a linear, time-invariant system whose unit-sample response $h[n]$ is shown below.

$$h[n] = \begin{cases} \left(\frac{1}{2}\right)^{n/2} & n = 0, 2, 4, 6, 8, \ldots, \infty \\ 0 & \text{otherwise} \end{cases}$$



Part a. Is it possible to represent this system with a finite number of poles?

Yes or **No**: Yes

If yes, enter the number of poles and list the pole locations below. If a pole is repeated k times, then enter that pole location k times. If there are more than 5 poles, enter just 5 of the pole locations. If there are fewer than 5 poles, leave the unused entries blank.

\# of poles: 2

locations:					
	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$			

If no, briefly explain why not.

$$H = 1 + \frac{1}{2}\mathcal{R}^2 + \frac{1}{4}\mathcal{R}^4 + \frac{1}{8}\mathcal{R}^6 + \frac{1}{16}\mathcal{R}^8 \cdots = \frac{1}{1 - \frac{1}{2}\mathcal{R}^2}$$

Substitute $\mathcal{R} \to \frac{1}{z}$:

$$H(z) = \frac{z^2}{z^2 - \frac{1}{2}} = \frac{z^2}{(z - \frac{1}{\sqrt{2}})(z + \frac{1}{\sqrt{2}})}$$

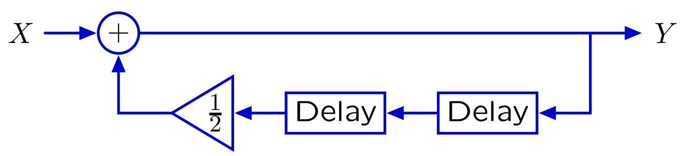
There are two poles:

$$z = \pm\frac{1}{\sqrt{2}} = \pm\frac{\sqrt{2}}{2}$$

Part b. Is it possible to implement this system with a finite number of adders, gains, and delays (and no other components)?

Yes or **No**: Yes

If yes, sketch a block diagram for the system in the following box.



If no, briefly explain why not.

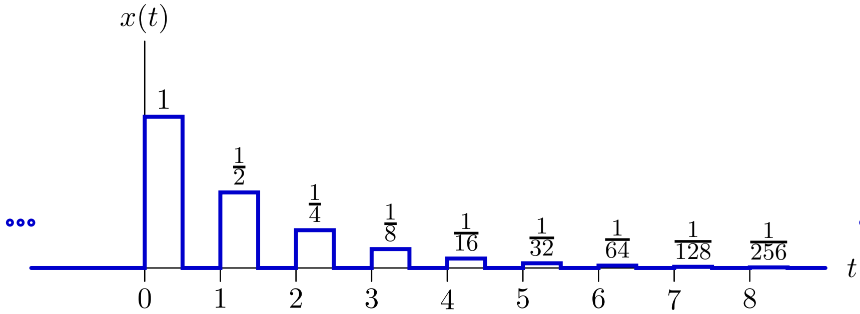
$$H = \frac{Y}{X} = 1 + \frac{1}{2}\mathcal{R}^2 + \frac{1}{4}\mathcal{R}^4 + \frac{1}{8}\mathcal{R}^6 + \frac{1}{16}\mathcal{R}^8 \cdots = \frac{1}{1 - \frac{1}{2}\mathcal{R}^2}$$

$$y[n] - \frac{1}{2}y[n-2] = x[n]$$

3. Laplace transform *[25 points]*

Determine the Laplace transform of $x(t)$ defined as follows.

$$x(t) = \begin{cases} 1 & 0 < t < 0.5 \\ 1/2 & 1 < t < 1.5 \\ 1/4 & 2 < t < 2.5 \\ 1/8 & 3 < t < 3.5 \\ 1/16 & 4 < t < 4.5 \\ \cdots & \\ 1/2^n & n < t < n+0.5 \\ \cdots & \\ 0 & \text{otherwise} \end{cases}$$



Enter a closed-form expression for the Laplace transform in the box below.

$X(s)=$ $\frac{1}{s}\left(\frac{1-e^{-s/2}}{1-\frac{1}{2}e^{-s}}\right)$

Enter the region of convergence (ROC) in the box below.

ROC= $\text{Re}(s) > -\ln 2$

$$x(t) = \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n\left(u(t-n)-u\left(t-n-\frac{1}{2}\right)\right)$$

$$X(s) = \int_{-\infty}^{\infty}\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n\left(u(t-n)-u\left(t-n-\frac{1}{2}\right)\right)e^{-st}dt$$

$$= \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n\int_{-\infty}^{\infty}\left(u(t-n)-u\left(t-n-\frac{1}{2}\right)\right)e^{-st}dt$$

$$= \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n\frac{1}{s}e^{-sn}\left(1-e^{-s/2}\right)$$

$$= \frac{1}{s}\left(1-e^{-s/2}\right)\sum_{n=0}^{\infty}\left(\frac{1}{2}e^{-s}\right)^n$$

$$= \frac{1}{s}\left(\frac{1-e^{-s/2}}{1-\frac{1}{2}e^{-s}}\right)$$

This transform converges if

$$\left|\frac{1}{2}e^{-s}\right| < 1$$

$$\left|e^{-\operatorname{Re}(s)}\right| < 2$$

$$\operatorname{Re}(s) > -\ln 2$$

4. Z transform *[25 points]*

Let $X(z)$ represent the Z transform of $x[n]$, and let $r_0 < |z| < r_1$ represent the region of convergence (ROC) of $X(z)$.

Let $Y(z)$ represent the Z transform of $y[n] = 2^n\,(u[n] + x[n])$ where $u[n]$ represents the unit-step signal.

Determine a closed-form expression for $Y(z)$ (which will depend on X) and enter the expression in the box below.

$Y(z)$: $\frac{z}{z-2} + X\left(\frac{z}{2}\right)$

Enter the region of convergence (ROC) for $Y(z)$ in the box below.

ROC: $\max(2, 2r_0) \; < \; |z| \; < \; 2r_1$

Let $y_1[n] = 2^n u[n]$. Then

$$Y_1(z) = \sum_{n=-\infty}^{\infty} 2^n u[n] z^{-n} = \sum_{n=0}^{\infty} \left(2z^{-1}\right)^n = \frac{1}{1-2z^{-1}} = \frac{z}{z-2} \quad \text{provided} \quad |z| > 2.$$

Let $y_2[n] = 2^n x[n]$. Then

$$Y_2(z) = \sum_{n=-\infty}^{\infty} 2^n x[n] z^{-n} = \sum_{n=-\infty}^{\infty} x[n] \left(\frac{z}{2}\right)^{-n} = X\left(\frac{z}{2}\right) \quad \text{provided} \quad 2r_0 < |z| < 2r_1.$$

$$Y(z) = Y_1(z) + Y_2(z) = \frac{z}{z-2} + X\left(\frac{z}{2}\right) \quad \text{provided} \quad \max(2, 2r_0) < |z| < 2r_1.$$

Worksheet (intentionally blank)

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