6.003 R

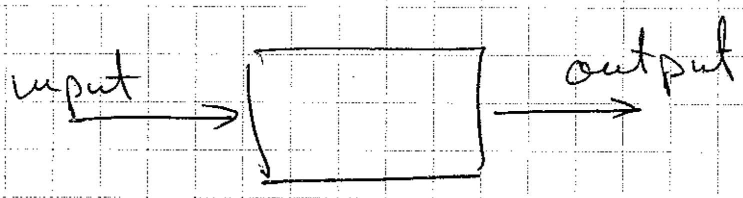
Feb 3, 2010

P. Hagelstein

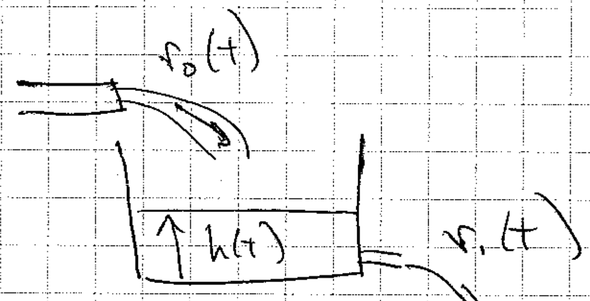
From lecture

Physical system $\longrightarrow$ Mathematical model

But now, the mathematical model we want to look at using signal and systems abstraction



Example physical system – filling a leaky bucket with water



$$v_0(t) = \frac{\text{Volume}}{\text{sec}}$$

Volume of water in bucket $V(t)$

$$V(t) = \text{Area} \cdot h(t)$$

$$\frac{d}{dt} V(t) = \frac{d}{dt} \text{Area}\, h(t) = v_o(t) - v_i(t)$$

$$\frac{d}{dt} h(t) = \frac{1}{\text{Area}} \left[v_o(t) - v_i(t) \right]$$

Relaxation time model

$$v_i(t) \sim h(t)$$

More complicated, so we would like a simpler way of thinking

let $v_i(t) = \text{Const} \; h(t)$

Fluid dynamics issues

$$\frac{d}{dt} h(t) = \frac{d}{dt} \frac{v_i(t)}{\text{const}} = \frac{1}{\text{Area}} \left[v_o(t) - v_i(t) \right]$$

$$\frac{d}{dt} v_i(t) = \frac{\text{Const}}{\text{area}} \left[v_o(t) - v_i(t) \right]$$

dimensions of $\frac{1}{\text{time}}$

$$\frac{1}{\tau} = \frac{\text{Const}}{\text{Area}}$$

Rewrite as

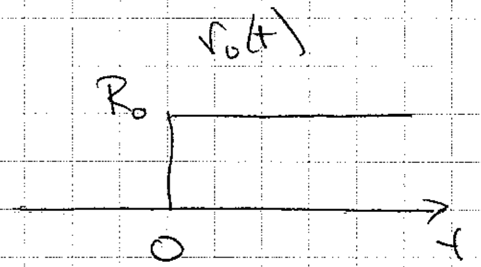
$$\frac{d}{dt} v_i(t) + \frac{v_i(t)}{\tau} = \frac{v_o(t)}{\tau}$$

Relaxation equation (is linear differential equation with constant coefficients)

<u>To solve:</u>

(a) write down answer by inspection
(b) homogeneous + inhomogeneous solutions
(c) general solution

Suppose that bucket is empty initially, and $v_o(t)$ turns on at $t=0$



what do we expect to happen?

Use intuition

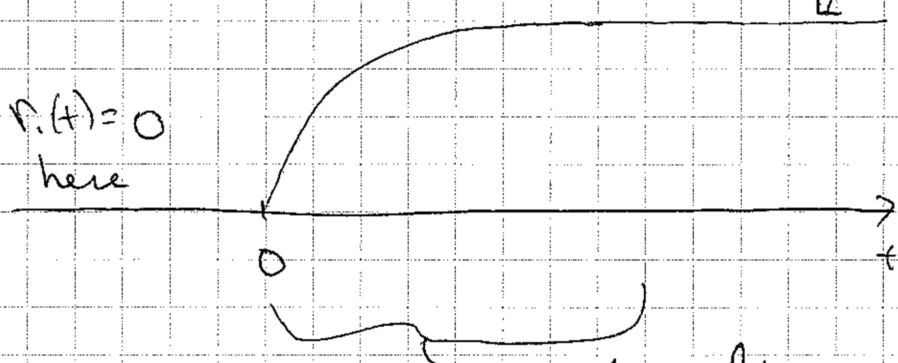
before $t=0$, no water in tank

large t, expect steady state solution

expect exponential decay from initial to final solution

$r_i(t) \to R_o$ here (What goes in must come out)

$r_i(t) = 0$ here



expect relaxation to steady state

Expect $r_i(t) = \begin{cases} 0 & t < 0 \\ R_o\left[1 - e^{-t/\tau}\right] & t > 0 \end{cases}$

(Guess)

Why... well

$$\frac{d}{dt} e^{-t/\tau} = -\frac{1}{\tau} e^{-t/\tau}$$

so $$\frac{d}{dt}\left(e^{-t/\tau}\right) + \frac{\left(e^{-t/\tau}\right)}{\tau} = 0$$

Differential equation, 1st order, constant coefficients

From math courses...

total solution = homogeneous solution + inhomogeneous solution

Inhomogeneous solution satisfies

$$\frac{d}{dt} v_1(t) + \frac{v_1(t)}{\tau} = \frac{v_0(t)}{\tau}$$

But maybe doesn't do so good with the boundary conditions. In our example, $v_0(t) = R_0$

$$\frac{d}{dt} v_1(t) + \frac{v_1(t)}{\tau} = \frac{R_0}{\tau}$$

Use $v_1(t) = R_0$ as inhomogeneous solution

Homogeneous solution satisfies

$$\frac{d}{dt} v_1(t) + \frac{v_1(t)}{\tau} = 0 \quad \text{(no source)}$$

Homogeneous solution is

$$v_1(t) = A e^{-t/\tau}$$

Total solution so far is

$$v_1(t) = R_0 + A e^{-t/\tau}$$

But we don't know what A is yet.
Find A from boundary conditions

$$v_1(0) = 0 = R_0 + A$$

use $A = -R_0$

$$v_1(t) = R_0 - R_0 e^{-t/\tau}$$
$$= R_0 \left[1 - e^{-t/\tau} \right] \qquad \text{for } t > 0$$

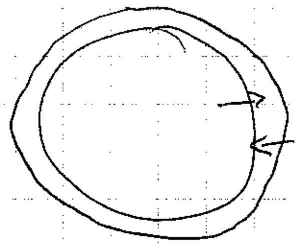
This agrees with our guess. Good.

Another example of a Relaxation model:

CO_2 in the atmosphere: natural sources

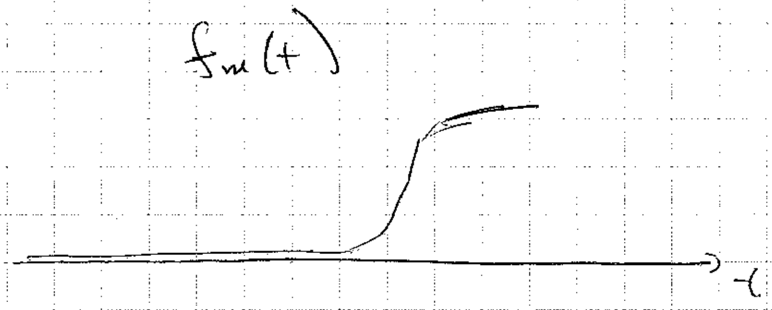
also sinks

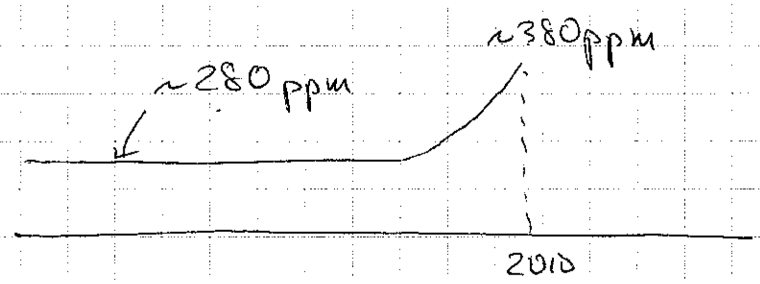
man-made sources



$$\frac{d}{dt} f(t) + \frac{f(t)}{\tau} = \frac{f_0}{\tau} + \frac{f_m(t)}{\tau}$$

So, how does it work?





Suppose we could stop putting CO_2 into the air? [What is τ? ~ 50-100 years]

<u>General solution</u>

Start with

$$\frac{d}{dt} v_1(t) + \frac{v_1(t)}{\tau} = \frac{v_0(t)}{\tau}$$

Then use math trick

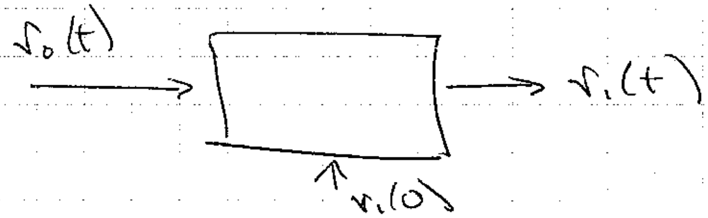
$$e^{-t/\tau} \frac{d}{dt}\left(e^{t/\tau} v_1(t)\right) = \frac{dv_1}{dt} + \frac{v_1}{\tau} = \frac{v_0(t)}{\tau}$$

$$\frac{d}{dt}\left(e^{t/\tau} v_1(t)\right) = e^{t/\tau} \frac{v_0(t)}{\tau}$$

Integrate and get

$$\left(e^{t/\tau} v_1(t)\right) - \left(v_1(0)\right) = \int_0^t e^{t'/\tau} \frac{v_0(t')}{\tau} dt'$$

$$v_1(t) = v_1(0)\, e^{-t/\tau} + \int_0^t e^{-\frac{(t-t')}{\tau}} \frac{v_0(t')}{\tau} dt'$$



Numerical Approach

$$\frac{d}{dt}y(t) \longrightarrow \frac{y_{n+1}-y_n}{\Delta t}$$

Forward Euler discretization when used in diff eq.

$$y(t) \longrightarrow y_n$$

$$\frac{dy}{dt} + \frac{y}{\tau} = s$$

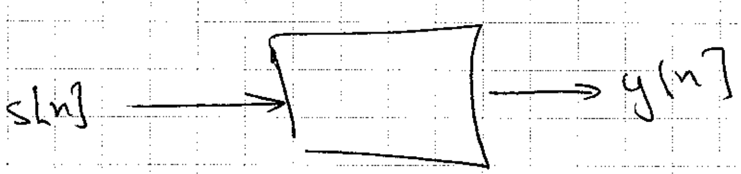
$$\frac{y_{n+1}-y_n}{\Delta t} + \frac{y_n}{\tau} = s_n$$

Numerical-type notation

$$y_{n+1} = y_n\left(1 \mp \frac{\Delta t}{\tau}\right) + \Delta t\, s_n$$

Using 6.003 language

$$y[n+1] = \left(1 \mp \frac{\Delta t}{\tau}\right) y[n] + \Delta t\, s[n]$$



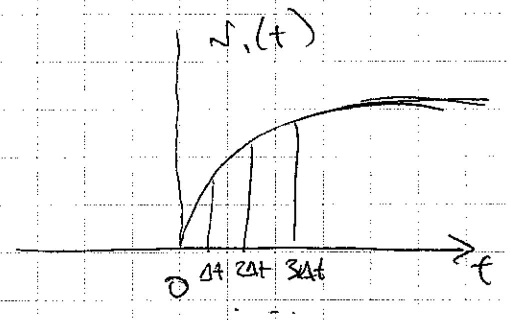
Our example

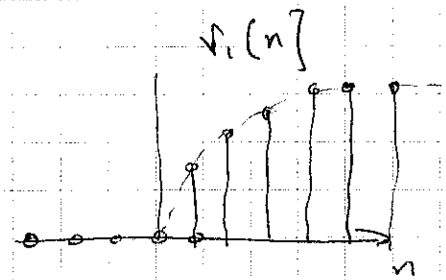
$$\frac{d}{dt} v_1(t) + \frac{v_1(t)}{\tau} = \frac{v_0(t)}{\tau}$$

$$\frac{v_1[n+1] - v_1[n]}{\Delta t} + \frac{v_1[n]}{\tau} = \frac{v_0[n]}{\tau}$$

$$v_1[n+1] = \left(1 - \frac{\Delta t}{\tau}\right) v_1[n] + \frac{\Delta t}{\tau} v_0[n]$$

$$v_0[n] \longrightarrow \boxed{} \longrightarrow v_1[n]$$



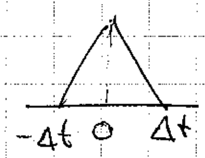


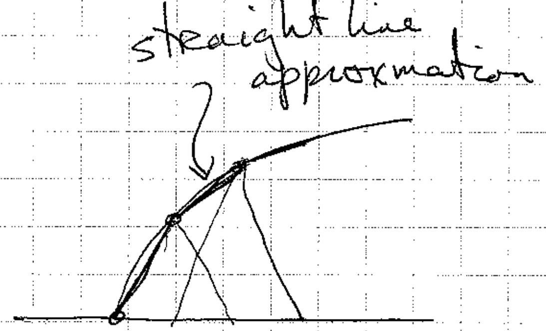
Can go back and forth

Linear approximation; suppose that you have a set of $v_1[n]$, and you would like to write down the approximation for $v_1(t)$?

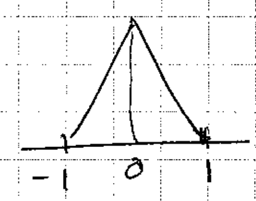
How to do it?

$$v_1(t) \approx \sum_n v_1[n]\,\phi(t-n\Delta t)$$





suppose we define $u(t)$



$$v_1(t) \approx \sum_n v_1[n]\,u\left(\frac{t}{\Delta t}-n\right)$$