6.003R Pt Hagelstein Feb. 17, 2010

Laplace transform – focus of next HW set, and introduction to be given in Thursday lecture.

$$\underbrace{X(s)}_{\text{Laplace Transform}} = \int_{-\infty}^{\infty} \underbrace{x(t)}_{\text{signal in time (CT)}} e^{-st}\, dt$$

Why are we interested?

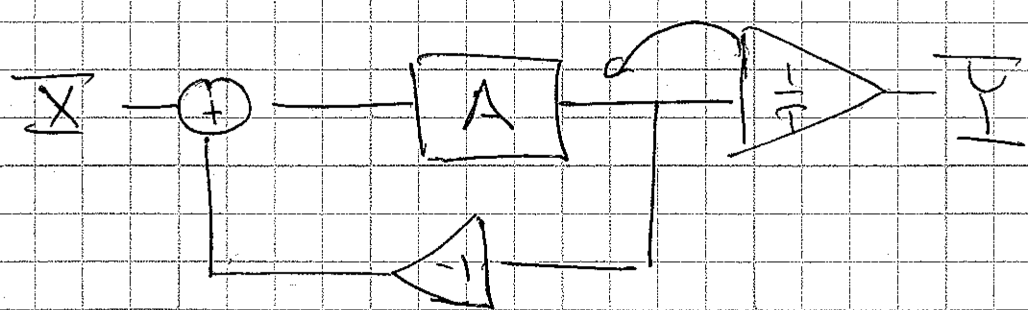
(a) Can relate to CT systems and A operator

(b) Can use it to solve diff eqns

(c) need to understand so we can do next HW set.

Example:

Suppose we have system:



We can find Y in terms of X

$$Y = \frac{1}{\tau} A \left[X - Y \right]$$

or

$$\left[1 + \frac{1}{\tau} A\right] Y = \frac{1}{\tau} A X$$

$$Y = \frac{\frac{1}{\tau} A X}{1 + \frac{1}{\tau} A}$$

$$\frac{Y}{X} = \frac{\frac{1}{\tau} A}{1 + \frac{1}{\tau} A}$$

Differential equation

Start with $\quad \frac{1}{A} Y + \frac{Y}{\tau} = \frac{X}{\tau}$

$$\frac{dy}{dt} + \frac{y}{\tau} = \frac{x}{\tau}$$

This is the ~~relaxa~~ relaxation time model we talked about in class a few weeks ago.

Use Laplace transform

$$\int_{-\infty}^{\infty} \underbrace{\frac{dy}{dt} e^{-st}}_{?} dt + \frac{1}{\tau}\int \underbrace{y e^{-st}}_{Y(s)} dt = \frac{1}{\tau}\int \underbrace{x e^{-st}}_{X(s)} dt$$

Use integration by parts

$$\int_a^b u\, dv = uv\Big|_a^b - \int_a^b v\, du$$

$$u = e^{-st} \qquad du = -s e^{-st} dt$$

$$dv = \left(\frac{dy}{dt}\right) dt \qquad v = y$$

$$\int_{-\infty}^{\infty} e^{-st} \frac{dy}{dt} dt = y(t) e^{-st}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} y\left(-s e^{-st}\right) dt$$

if $y(t) e^{-st}\Big|_{-\infty}^{\infty} = 0$, then

$$\int_{-\infty}^{\infty} e^{-st} \frac{dy}{dt} dt = s\int_{-\infty}^{\infty} y e^{-st} dt$$

$$= sY(s)$$

So

$$sY(s) + \frac{Y(s)}{\tau} = \frac{X(s)}{\tau}$$

$$\left(s + \frac{1}{\tau}\right) Y(s) = \frac{X(s)}{\tau}$$

$$\boxed{Y(s) = \frac{\frac{1}{\tau}}{s + \frac{1}{\tau}} X(s)}$$

Compare with operator version

$$Y(A) = \frac{\frac{1}{\tau}A}{1 + \frac{1}{\tau}A} X$$

$$= \frac{\frac{1}{\tau}}{\frac{1}{A} + \frac{1}{\tau}} X$$

Interesting...

$$s \Longleftrightarrow \frac{1}{A}$$

Provides connection between operator algebra and transform!

$\Rightarrow$ Can use operator methods we already know to work ~~B~~ Laplace transforms

$\Rightarrow$ Different ways to work and think about problems!

To use Laplace transforms, we need to get familiar with some specific examples.

$x(t) = \delta(t)$

$$X(s) = \int_{-\infty}^{\infty} \delta(t)\, e^{-st}\, dt$$

$$= 1$$

independent of s. Cool.

$x(t) = \delta(t-1)$

$$X(s) = \int_{-\infty}^{\infty} \delta(t-1)\, e^{-st}\, dt$$

$$= e^{-s}$$

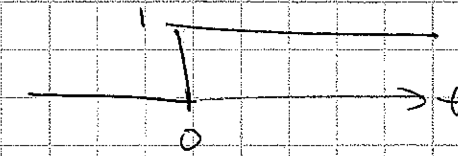
Hmmm... an offset in t looks like it contributes e^{-s} somehow...

$x(t) = \delta(t+1)$ as a check...

$$X(s) = \int_{-\infty}^{\infty} \delta(t+1)\, e^{-st}\, dt = e^{s}$$

ok so $\delta(t-T) \longleftrightarrow e^{-sT}$

$x(t) = u(t)$



$$X(s) = \int_{-\infty}^{\infty} u(t)\, e^{-st}\, dt$$

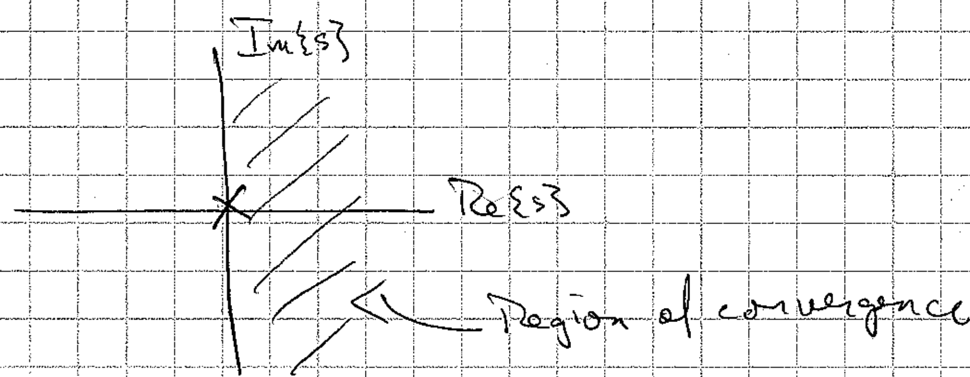
$$= \int_{0}^{\infty} e^{-st}\, dt \quad \left(= -\frac{e^{-st}}{s}\Big|_0^{\infty} \right)$$

$$= \frac{1}{s}$$

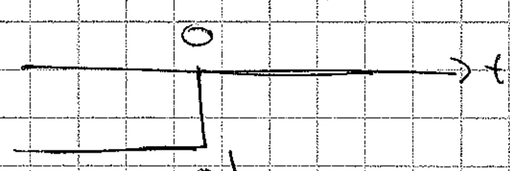
Hmmm... we get a pole. Wonder if that is important?

Does integral converge for $s = 1$ ... yes

for $s = -1$ -- no



$x(t) = -u(-t)$



$$X(s) = \int_{-\infty}^{\infty} -u(-t)\, e^{-st}\, dt$$

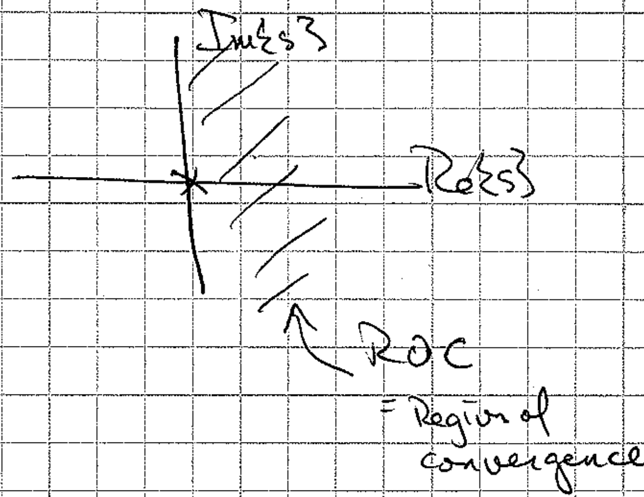
$$= -\int_{-\infty}^{0} e^{-st}\, dt = \frac{1}{s} e^{-st}\Big|_{-\infty}^{0} = \frac{1}{s} \quad \text{if } e^{-s(\infty)} = 0$$

Hmmm... same Laplace transform, but different ROC

Think about it

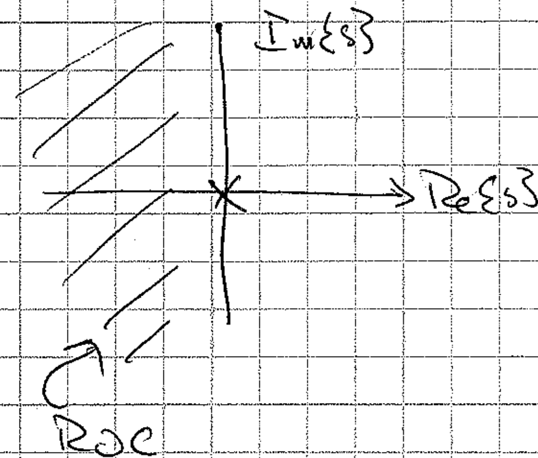
$x(t) = u(t)$

$X(s) = \frac{1}{s}$



$x(t) = -u(-t)$

$X(s) = \frac{1}{s}$



OK... seems to make sense...

Try $x(t) = t\,u(t)$

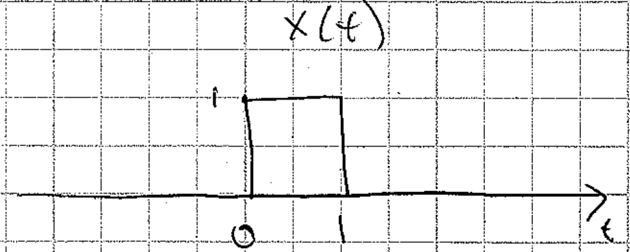
$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

$$= \int_{0}^{\infty} t\, e^{-st}\, dt$$

$$= -\frac{d}{ds}\int_{0}^{\infty} e^{-st}\, dt = -\frac{d}{ds}\frac{1}{s} = \frac{1}{s^2}$$

Expect region of convergence to be $Re\{s\} > 0$ as before

More complicated examples



Two approaches possible... (at least)

Integral: $$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st}\, dt = \int_0^1 e^{-st}\, dt$$

$$= -\frac{e^{-st}}{s}\Bigg|_0^1 = -\frac{e^{-s}}{s} + \frac{1}{s}$$

$$= \frac{1 - e^{-s}}{s}$$

Superposition:

$$x(t) = u(t) - u(t-1)$$

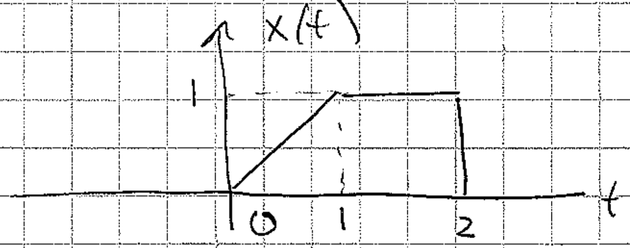
$$\downarrow \qquad\qquad \downarrow$$

$$X(s) = \frac{1}{s} - \frac{e^{-s}}{s}$$

$\frac{1}{s} \leftrightarrow$ step

$e^{-s} \leftrightarrow$ delay

Another example



Think superposition:

Ramp up at $t=0$ + Ramp down at $t=1$ + step down at $t=2$

$$\bar{X}(s) = \frac{1}{s^2} - \frac{1}{s^2}e^{-s} - \frac{1}{s}e^{-2s}$$

Work out the algebra to check..

$$\bar{X}(s) = \int_{-\infty}^{\infty} x(t) e^{-st}\,dt$$

$$= \int_0^1 t\,e^{-st}\,dt + \int_1^2 e^{-st}\,dt$$

$$= -\frac{d}{ds}\int_0^1 e^{-st}\,dt - \left.\frac{e^{-st}}{s}\right|_1^2$$

$$= -\frac{d}{ds}\left(-\left.\frac{e^{-st}}{s}\right|_0^1\right) - \frac{e^{-2s}-e^{-s}}{s}$$

$$= \frac{d}{ds}\left(\frac{e^{-s}-1}{s}\right) - \frac{e^{-2s}-e^{-s}}{s}$$

$$= -\left(\frac{e^{-s}-1}{s^2}\right) - \frac{e^{-s}}{s} + \frac{e^{-s}}{s} - \frac{e^{-2s}}{s}$$

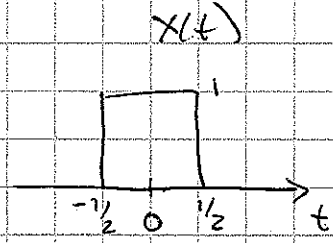
$$= \frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s}$$

OK - math seems to give the same answer. Wow!
it works (applause... standing ovation)

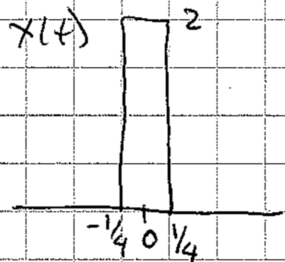
TOPS FORM 3504 I MADE IN U.S.A.

Thinking about impulse functions

From class



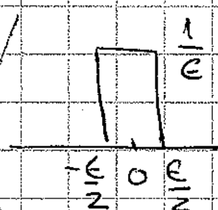
$$\int_{-\infty}^{\infty} x(t)\,dt = 1$$



$$\int_{-\infty}^{\infty} x(t)\,dt = 1$$

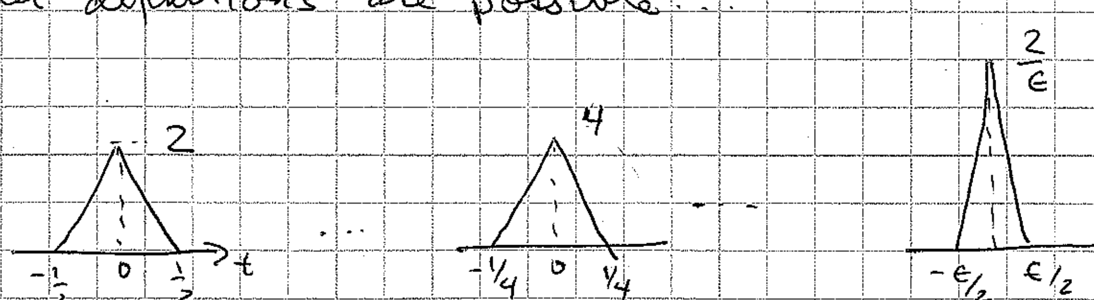
If we want to keep area constant, but make the signal shorter in duration, then the amplitude gets bigger

$$\delta(t) = \lim_{\epsilon \to 0} \Big($$



$$\Big)$$

Other definitions are possible...



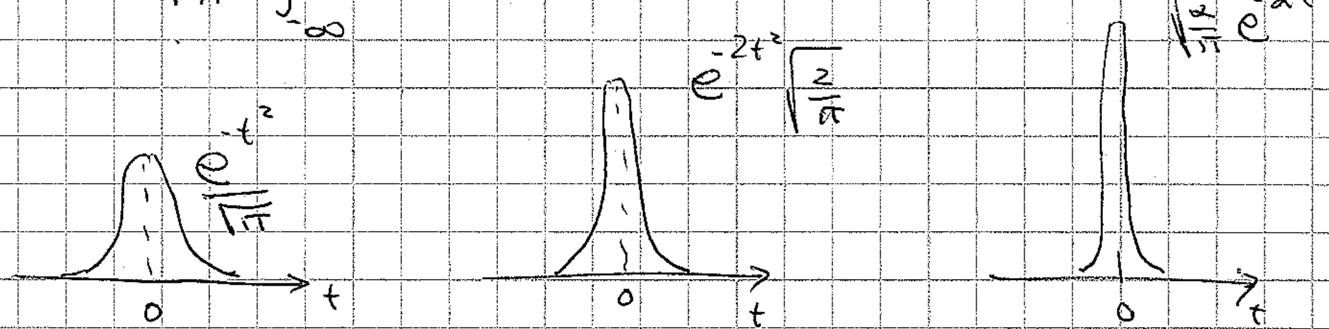
$$\int_{-\infty}^{\infty} x(t)\,dt = 1$$

A

Also possible to use smooth function

$$\int_{-\infty}^{\infty} e^{-\alpha t^2}\, dt = \sqrt{\frac{\pi}{\alpha}}$$

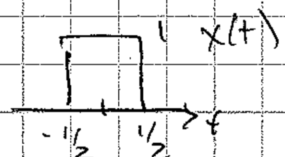
So $$\sqrt{\frac{\alpha}{\pi}} \int_{-\infty}^{\infty} e^{-\alpha t^2}\, dt = 1$$



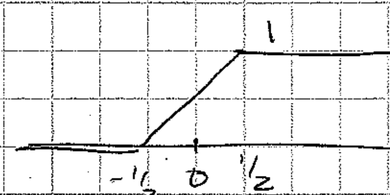
$$\delta(t) = \lim_{\alpha \to \infty} \left(\sqrt{\frac{\alpha}{\pi}}\, e^{-\alpha t^2} \right)$$

$$u(t) = \int_{-\infty}^{t} \delta(t')\, dt'$$

Also get smoothed versions by doing integrals of approximate $\delta(t)$ functions; can then take limit.

Start with 

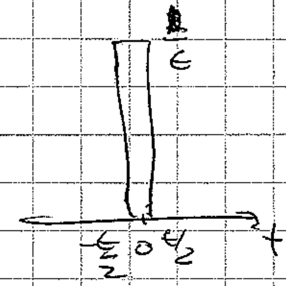
$$\int_{-\infty}^{t} x(t')\, dt' = \begin{cases} 0 & t < -\frac{1}{2} \\ t + \frac{1}{2} & -\frac{1}{2} < t < \frac{1}{2} \\ 1 & t > \frac{1}{2} \end{cases}$$



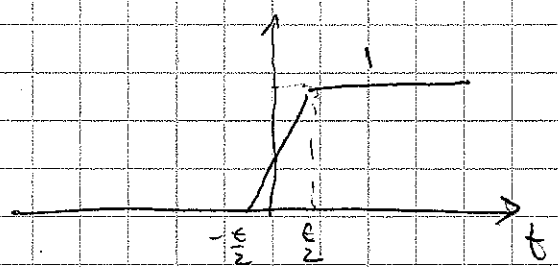
(B)

Suppose then

$$x(t) = \begin{cases} 0 & \cancel{t} < -\frac{\epsilon}{2} \\ \frac{1}{\epsilon} & -\frac{\epsilon}{2} < x < \frac{\epsilon}{2} \\ 0 & \cancel{t} > \frac{\epsilon}{2} \end{cases}$$



$$u(t) \approx \int_{-\infty}^{t} x(t)\,dt = \begin{cases} 0 & t < -\frac{\epsilon}{2} \\ \left(\frac{t+\frac{\epsilon}{2}}{\epsilon}\right) & -\frac{\epsilon}{2} < t < \frac{\epsilon}{2} \\ 1 & \frac{\epsilon}{2} < t \end{cases}$$



get $u(t)$ from limit $\epsilon \to 0$

(c)

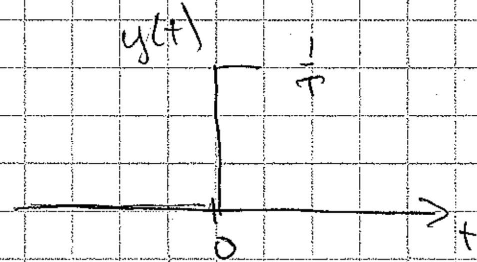
$$\int_{-\epsilon}^{\epsilon} \frac{x(t')}{\tau}\, dt' = \frac{1}{\tau}$$

get $$\int_{-\epsilon}^{\epsilon} \frac{dy}{dt} + \frac{y}{\tau}\, dt' = \int_{-\epsilon}^{\epsilon} \frac{x(t')}{\tau}\, dt'$$

$$\downarrow$$

$$\underbrace{y(\epsilon)}_{\text{"big"}} - \underbrace{y(-\epsilon)}_{0} + \underbrace{\frac{\epsilon}{2\tau}}_{\text{very small}}\left[y(\epsilon) + \underbrace{y(-\epsilon)}_{0}\right] \approx \underbrace{\frac{1}{\tau}}_{\text{"big"}}$$

So $y(\epsilon) \approx \frac{1}{\tau}$



expect a step near $t = 0$

From there, we can solve a boundary condition problem.

for $t > 0$

$$\frac{dy}{dt} + \frac{y}{\tau} = 0$$

$$y(t) = Ae^{-t/\tau}$$

$$\Rightarrow y(t) = \begin{cases} 0 & t < 0 \\ \frac{e^{-t/\tau}}{\tau} & t \geq 0 \end{cases}$$

or $= u(t)\frac{e^{-t/\tau}}{\tau}$

E Choose $y(0^+) = \frac{1}{\tau} \Rightarrow A = \frac{1}{\tau}$

Thinking about differential eqns

$$\frac{dy}{dt} + \frac{y}{\tau} = \frac{x}{\tau}$$

Suppose $x(t) = \delta(t)$

what do we expect for $y(t)$ near $t=0$?

Does $y(t)$ go like $\delta(t)$?

Does $\frac{dy}{dt}$ go like $\delta(t)$?

if so, then $\frac{dy}{dt} \approx \frac{\delta(t)}{\tau}$ small t

$$y(t) \approx \int_{-\infty}^{t} \frac{\delta(t'')}{\tau}\, dt'$$

$$\approx \frac{1}{\tau}\, u(t) \quad \text{small } t$$

$$\int_{-\epsilon}^{\epsilon} \frac{dy}{dt} + \frac{y}{\tau}\, dt' = \int_{-\epsilon}^{\epsilon} \frac{x}{\tau}\, dt'$$

$$\int_{-\epsilon}^{\epsilon} \frac{dy}{dt}\, dt' = \int_{y(-\epsilon)}^{y(\epsilon)} dy = y(\epsilon) - y(-\epsilon)$$

$$\int_{-\epsilon}^{\epsilon} \frac{y}{\tau}\, dt' \approx \frac{\epsilon}{2}\,\frac{1}{\tau}\left(y(\epsilon) + y(\epsilon')\right)$$

Another try

$$\frac{dy}{dt} + \frac{y}{T} = \frac{x}{T} \qquad x(t) = u(t)$$

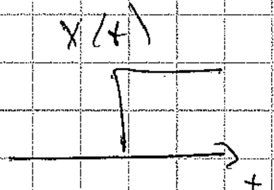
Try same argument

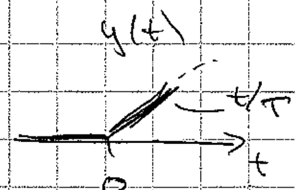
small near 0 if $y(0^-) = 0$

$$\left(\frac{dy}{dt}\right)_{\substack{\text{near}\\ t=0}} + \cancel{\frac{y}{T}} = \frac{u(t)}{T}$$

$$y(t) \approx \int_{-\infty}^{t} \frac{u(t')}{T}\, dt'$$

$$= \frac{1}{T}\int_0^t dt' = \frac{t}{T} \qquad \text{small } t,\ t>0$$





What happens later on?

$$y(t) = y_{hom} + y_{inhom}$$

$y_{hom} = Ae^{-t/T}$ $\qquad$ $y_{inhom} \to y = 1$

$$y(0) = 0 = A + 1 \Rightarrow A = -1$$

$$y(t) = 1 - e^{-t/T} \longrightarrow t/T \quad \text{small } t$$

for $t > 0$

$$y(t) = u(t)\left[1 - e^{-t/T}\right]$$

General solution

Start with

$$\frac{d}{dt}y + \frac{y}{\tau} = \frac{x}{\tau}$$

Then use math trick

$$e^{-t/\tau}\frac{d}{dt}\left(e^{t/\tau}y\right) = e^{-t/\tau}\left(e^{t/\tau}\frac{dy}{dt} + e^{t/\tau}\frac{y}{\tau}\right)$$

$$= \frac{dy}{dt} + \frac{y}{\tau}$$

So

$$e^{-t/\tau}\frac{d}{dt}\left(e^{t/\tau}y\right) = \frac{x}{\tau}$$

or

$$\frac{d}{dt}\left(e^{t/\tau}y\right) = e^{t/\tau}\frac{x}{\tau}$$

Then

$$\int_{-\infty}^{t}\frac{d}{dt'}\left(e^{t'/\tau}y\right)dt' = \int_{-\infty}^{t} e^{t'/\tau}\frac{x}{\tau}\,dt'$$

F

$$e^{t/\tau} y(t) - \cancel{e^{-\infty/\tau} y(-\infty)}^{\,0} = \int_{-\infty}^{t} e^{t'/\tau} \frac{x(t')}{\tau} dt'$$

$$y(t) = \int_{-\infty}^{t} \frac{e^{-(t-t')/\tau}}{\tau} x(t')\, dt'$$

Check... let $x(t) = \delta(t)$

$$y(t) = \int_{-\infty}^{t} \frac{e^{-\frac{(t-t')}{\tau}}}{\tau} \delta(t')\, dt'$$

$$= \begin{cases} \dfrac{e^{-t/\tau}}{\tau} & t > 0 \\ 0 & t < 0 \end{cases}$$

$$y(t) = u(t) \frac{e^{-t/\tau}}{\tau}$$ same as we got before

6

TOPS FORM 35041 MADE IN U.S.A.

$$y(t) = \int_{-\infty}^{t} h(t-t')\, x(t')\, dt'$$

$$h(t) = \int_{-\infty}^{t} h(t-t')\, \delta(t')\, dt'$$

OK, suppose now that we put in $\frac{d}{dt}\delta(t)$

$$? = \int_{-\infty}^{t} h(t-t')\, \frac{d}{dt'}\delta(t')\, dt'$$

$$\int_a^b u\, dv = uv\Big|_a^b - \int v\, du$$

$$u(t') = h(t-t') \qquad du = \frac{dh}{dt'}\, dt'$$

$$dv = \frac{d}{dt'}\delta(t') \qquad \cancel{d}v = \delta(t')$$

$$\int_{-\infty}^{t} h(t-t')\, \frac{d}{dt'}\delta(t')\, dt' = h(t-t')\,\delta(t')\Big|_{-\infty}^{\infty}$$

$$- \int_{-\infty}^{t} \frac{dh(t-t')}{dt'}\, \delta(t')\, dt'$$

$$= \cancel{\frac{dh(t-t')}{dt}} + \frac{dh}{dt}$$