Example

Peter Hagelstein

$x(t)$ is odd

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

What can we say about $X(s)$?

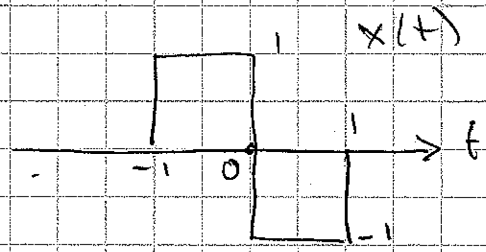
$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt = -\int_{-\infty}^{\infty} x(-t) e^{-st} dt$$

let $t' = -t \qquad dt' = -dt$

$$= -\int_{-\infty}^{\infty} x(t') e^{st'} dt' = -X(-s)$$

From exchanging limits

Interesting: lets check



$$X(s) = \frac{1}{s} e^{s} - \frac{2}{s} + \frac{1}{s} e^{-s}$$

$$= \frac{2}{s} \left[\cosh s - 1 \right]$$

$$X(-s) = -\frac{2}{s} \left[\cosh s - 1 \right] = -X(s)$$ it seems to work!

Example

Suppose that

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

Find $\int_{-\infty}^{\infty} x(t)\, dt$ in terms of $X(s)$

$$X(0) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\Big|_{s=0} dt = \int_{-\infty}^{\infty} x(t)\, dt$$

Find $x(\infty) - x(-\infty)$ in terms of $X(s)$

This one is less obvious

Note that $$\int_{-\infty}^{\infty} \left(\frac{dx}{dt}\right) dt = \int_{x(-\infty)}^{x(\infty)} dx = x \Big|_{x(-\infty)}^{x(\infty)}$$

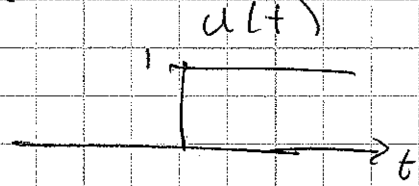
$$= x(\infty) - x(-\infty)$$

But $$\int_{-\infty}^{\infty} \left(\frac{dx}{dt}\right) dt = \lim_{s \to 0} \int_{-\infty}^{\infty} \frac{dx}{dt} e^{-st}\, dt$$

$$= \lim_{s \to 0} \left(s X(s) \right)$$

Check to see if it works.

let $x(t) = u(t)$



$$X(s) = \int_{-\infty}^{\infty} e^{-st}\, u(t)\, dt$$

$$= \int_0^{\infty} e^{-st}\, dt = \frac{1}{s} \qquad \mathrm{Re}\{s\} > 0$$

$$\lim_{s \to 0} \left(s \cdot \frac{1}{s} \right) = 1$$

$$x(\infty) - x(-\infty) = 1 \qquad \text{it seems to work}$$

OK. what about $\left.\frac{dx}{dt}\right|_{\infty} - \left.\frac{dx}{dt}\right|_{-\infty}$

$$\int_{-\infty}^{\infty} \left(\frac{d^2x}{dt^2} \right) dt = \left.\frac{dx}{dt}\right|_{-\infty}^{\infty} = \left.\frac{dx}{dt}\right|_{\infty} - \left.\frac{dx}{dt}\right|_{-\infty}$$

$$\left.\frac{dx}{dt}\right|_{\infty} - \left.\frac{dx}{dt}\right|_{-\infty} = \lim_{s \to 0} \left[s^2 X(s) \right]$$

Check: $x(t) = t\,u(t) \longleftrightarrow X(s) = \frac{1}{s^2} \quad \mathrm{Re}\{s\} > 0$

$$\lim_{s \to 0} \left(s^2 \cdot \frac{1}{s^2} \right) = 1 \qquad \left.\frac{dx}{dt}\right|_{\infty} = 1 \quad \left.\frac{dx}{dt}\right|_{-\infty} = 0$$

it works!

How might we get at $x(0)$?

$$\bar{X}(s) = \int_{-\infty}^{\infty} e^{-st} x(t)\,dt$$

Why not look at large s Real for signals that are of the form $u(t)[\cdots]$

$$\lim_{s\to\infty} \bar{X}(s) = \lim_{s\to\infty}\int_{-\infty}^{\infty} e^{-st} x(t)\,dt$$

$$= \lim_{s\to\infty} \int_{0}^{\infty} e^{-st} x(t)\,dt$$ since we assumed $x(t)$ is $u(t)[\cdots]$

$$\approx \lim_{s\to\infty} x(0)\int_{0}^{\infty} e^{-st}\,dt$$

$$= \lim_{s\to\infty} x(0)\,\frac{1}{s}$$

So maybe

$$x(0^+) = \lim_{s\to\infty} \left[s\,\bar{X}(s)\right]$$

Check...

$x(t) = u(t) \qquad \bar{X}(s) = \frac{1}{s} \quad \mathrm{Re}\{s\} > 0$

$$\lim_{s\to\infty}\left[s \cdot \frac{1}{s}\right] = 1 = x(0^+)$$

seems to work

Can ~~wee~~ get derivative also?

$$\lim_{s\to\infty}\left(s\,\bar{X}(s)\right) = \lim_{s\to\infty}\int_{-\infty}^{\infty}\frac{dx}{dt}e^{-st}\,dt = \lim_{s\to\infty}\int_{0}^{\infty}\frac{dx}{dt}e^{-st}\,dt$$

$$\longrightarrow \left(\frac{dx}{dt}\right)_0\int_0^{\infty}e^{-st}\,dt = \frac{1}{s}\left.\frac{dx}{dt}\right|_0$$

$$\left.\frac{dx}{dt}\right|_0 = \lim_{s\to\infty}\left[s^2\,\bar{X}(s)\right]$$

Cool!

Can we get something from $\lim\limits_{s\to\infty}\bar{X}(s)$?

Start with

$$\frac{\bar{X}(s)}{s} = \int_{-\infty}^{\infty} y(t)e^{-st}\,dt$$

$$y(t) = \int_{-\infty}^{t} x(t')\,dt'$$

$$\lim_{s\to\infty}\frac{\bar{X}(s)}{s} = y(0)\,\frac{1}{s} \qquad y(0) = \lim_{s\to\infty}\bar{X}(s)$$

5

Try it out...

$$x(t) = \delta(t) \quad \longleftrightarrow \quad X(s) = 1$$

$$y(t) = \int_{-\infty}^{t} x(t')\,dt' = u(t)$$

$$y(0^+) = \lim_{s \to \infty} X(s) = 1 \quad \underline{\underline{ok}}$$

Example

$$x(t) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{st}\, ds$$

Find: $\frac{dx}{dt}$:

$$\frac{dx}{dt} = \frac{1}{2\pi j}\frac{d}{dt}\int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{st}\, ds$$

$$= \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} \frac{d}{dt}\left(X(s) e^{st}\right) ds$$

$$= \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} X(s)\, s\, e^{st}\, ds$$

So if

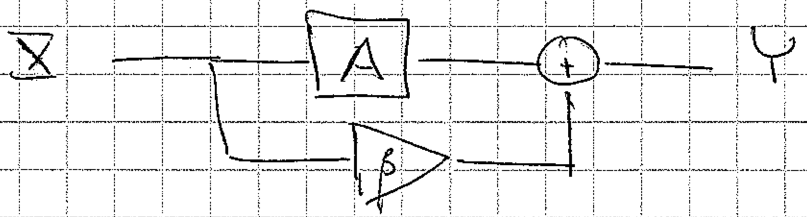
$$x(t) \longleftrightarrow X(s)$$

Then

$$\frac{dx}{dt} \longleftrightarrow sX(s)$$

we found this by integration by parts last time

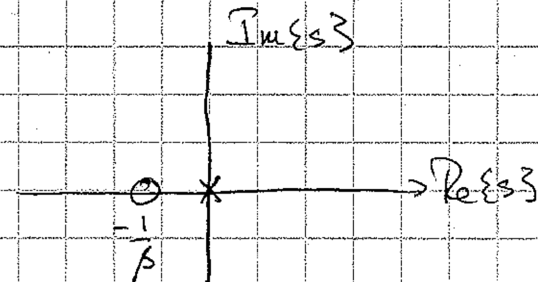
Connection between circuits, eqns, transforms



$$Y = (A + \beta) X$$

$$Y(s) = \left(\frac{1}{s} + \beta\right) X(s) = \left(\frac{\beta s + 1}{s}\right) X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\beta s + 1}{s}$$



Diff eqn

Start with

$$Y(s) = \left(\frac{\beta s + 1}{s}\right) X(s)$$

$$s Y(s) = \beta s X(s) + X(s)$$

$$\frac{dy}{dt} = \beta \frac{dx}{dt} + x(t)$$

Impulse response

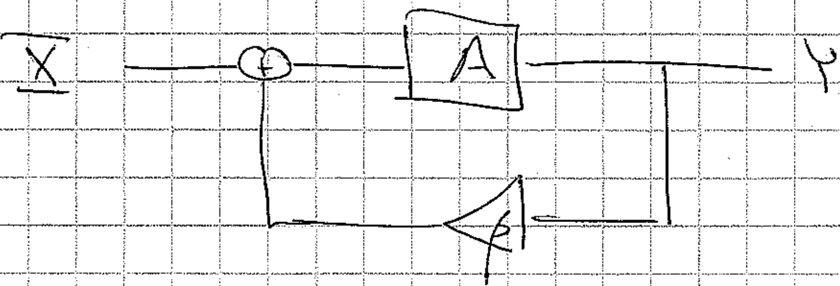
$$H(s) = \frac{\beta s + 1}{s} = \beta + \frac{1}{s}$$

$$h(t) = \beta \delta(t) + u(t)$$

$$\frac{dy}{dt} = \beta \frac{dx}{dt} + x \Longrightarrow y = \beta x + \int_{-\infty}^{t} x + \text{Const}$$

for $x = \delta(t)$, $y(0^-) = 0$ $\quad y = \beta \delta + u$

Try another one



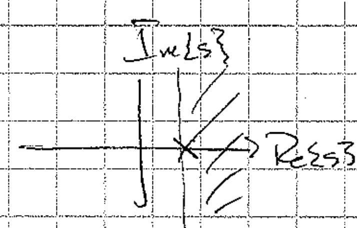
$$Y = A(X + \beta Y)$$

$$(1 - \beta A)\,Y = A X$$

$$Y = \frac{A}{1-\beta A} X$$

$$Y(s) = \frac{1}{s-\beta} X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s-\beta}$$



Diff eqn

$$Y(s) = \frac{1}{s-\beta} X(s)$$

$$(s-\beta)\,Y(s) = X(s)$$

$$\frac{dy}{dt} - \beta y = x(t)$$

Impulse response

$$H(s) = \frac{1}{s-\beta} \Rightarrow h(t) = u(t)\,e^{\beta t}$$

$$\frac{dy}{dt} - \beta y = x(t)$$

$$e^{\beta t}\frac{d}{dt}\left(e^{-\beta t}y\right) = x$$

$$\frac{d}{dt}\left(e^{-\beta t}y\right) = e^{-\beta t}x$$

$$\int_{-\infty}^{t}\frac{d}{dt'}\left(e^{-\beta t'}y\right)dt' = \int_{-\infty}^{t} e^{-\beta t'}x(t')\,dt'$$

$$e^{-\beta t}y(t)\Big|_{-\infty}^{t} = \int_{-\infty}^{t} e^{-\beta t'}x(t')\,dt'$$

$$e^{-\beta t}y(t) - e^{\beta\infty}y(-\infty) = \int_{-\infty}^{t} e^{-\beta t'}x(t')\,dt'$$

Assume $y(-\infty) = 0$

$$y(t) = e^{\beta t}\int_{-\infty}^{t} e^{-\beta t'}x(t')\,dt'$$

$$= \int_{-\infty}^{t} e^{\beta(t-t')}x(t')\,dt'$$

if $x(t) = \delta(t)$

$$y(t) = e^{\beta t}u(t)$$

$$\beta s \bar{X}(s) \longleftrightarrow \beta \frac{dx}{dt}$$

So, we expect

$$h(t) = \beta \frac{d}{dt}\left(u(t) e^{\beta t} \right)$$

$$= \beta \delta(t) e^{\beta t} + u(t) \beta^2 e^{\beta t}$$

$$= \beta \delta(t) + u(t) \beta^2 e^{\beta t}$$

<u>Solve using diff equ</u>

$$\frac{dy}{dt} - \beta y = \beta \frac{dx}{dt}$$

$$e^{\beta t} \frac{d}{dt}\left(e^{-\beta t} y\right) = \beta \frac{dx}{dt}$$

$$\frac{d}{dt}\left(e^{-\beta t} y\right) = e^{-\beta t} \beta \frac{dx}{dt}$$

$$\int_{-\infty}^{t} \frac{d}{dt'}\left(e^{-\beta t'} y(t')\right) dt' = \int_{-\infty}^{t} e^{-\beta t'} \beta \frac{dx}{dt'} dt'$$

$$e^{-\beta t} y(t) \Big|_{-\infty}^{t} = e^{-\beta t} y(t) = \int_{-\infty}^{t} e^{-\beta t'} \beta \frac{dx}{dt'} dt'$$

assuming $y(t) = 0$ for $t < 0$

$$y(t) = \int_{-\infty}^{t} e^{\beta(t-t')} \beta \frac{dx}{dt'} \, dt'$$

OK, let $x = \delta(t)$...

Not so easy to integrate... use integration by parts

$$y(t) = \int_{-\infty}^{t} h(t-t') \frac{dx}{dt'} \, dt'$$

$$\int_a^b u \, dv = uv \Big|_a^b - \int_a^b v \, du$$

$$u = h(t-t') \qquad du = \frac{d}{dt'} h(t-t') \, dt'$$

$$dv = \frac{dx}{dt'} dt' \qquad v = x$$

$$\int_{-\infty}^{t} h(t-t') \frac{dx}{dt'} dt' = x(t') h(t-t') \Big|_{-\infty}^{t} - \int_{-\infty}^{t} \frac{d}{dt'} h(t-t') \, x(t') \, dt'$$

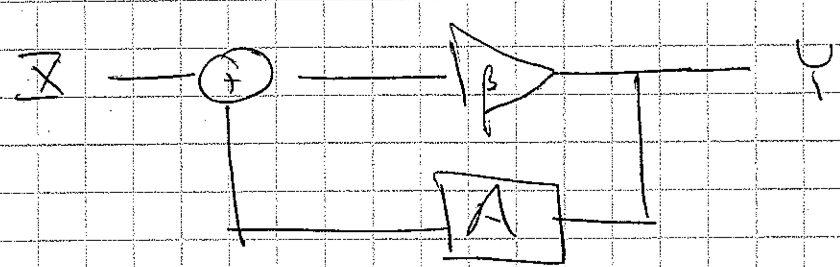
$$= \delta(t) \beta + \int_{-\infty}^{t} \frac{d}{dt} h(t-t') \, x(t') \, dt'$$

changed variable

$$= \delta(t) \beta + \frac{dh}{dt}$$

$$= \beta \delta(t) + \frac{d}{dt} \beta e^{\beta t} \qquad = \beta \delta(t) + \beta^2 e^{\beta t} u(t)$$

$$= \beta \delta(t) + \beta^2 e^{\beta t}$$ for $t > 0$



$$Y = \beta(X + AY)$$

$$(1 - \beta A)Y = \beta X$$

$$Y = \frac{\beta}{1-\beta A} X$$

$$Y(s) = \frac{\beta}{1-\beta/s} X(s) = \frac{s\beta}{s-\beta} X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\beta s}{s-\beta}$$

Diff eqn

$$Y(s) = \frac{s\beta}{s-\beta} X(s)$$

$$(s-\beta)Y(s) = s\beta X(s)$$

$$\frac{dy}{dt} - \beta y = \beta \frac{dx}{dt}$$

Impulse response $H(s) = \dfrac{\beta s}{s-\beta}$

Hmmm, know that if we had $\frac{1}{s-\beta}$, then it

~~7~~ → 13 would be $u(t)e^{\beta t}$. So what does βs do?