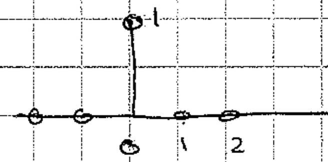
6.728R   P. Hagelstein   Feb 24, 2010

Z-Transforms, lets start simple

$x[n] = \delta[n]$

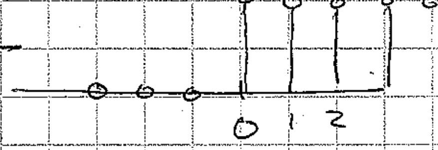


$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$= 1$$

OK ...

Try $x[n] = u[n]$



$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{n=0}^{\infty} z^{-n}$$

$$= \frac{1}{1 - 1/z} = \frac{z}{z-1} \qquad \text{ROC } \left|\tfrac{1}{z}\right| < 1$$

or $|z| > 1$

Thinking... Accumulator functionality?

Suppose $x[n] \longleftrightarrow X(z)$

what is $Y_1(z) = \dfrac{z}{z-1} X(z)$

Think in terms of operators

$$Y_1 = \frac{1}{1 - 1/z} X = \frac{1}{1-R} X$$

$$y[n] = (1 + R + R^2 + R^3 + \cdots)\, x[n]$$

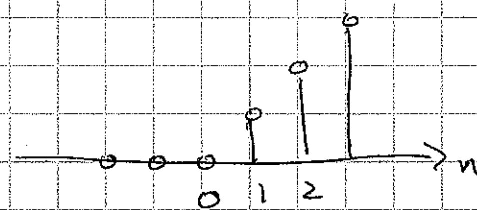
TOPS FORM 35041 MADE IN U.S.A.

Analogy to integration in CT problem

$$y[n] = \sum_{m=-\infty}^{n} x[m]$$

Ramp

$$x[n] = n\,u[n]$$



Think in terms of accumulator

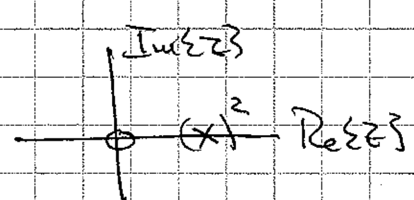
$$x[n] = \sum_{m=-\infty}^{n} u[m-1]$$

$$x[-1] = 0$$
$$x[0] = 0$$
$$x[1] = 0 + 1 = 1$$
$$x[2] = 0 + 1 + 1 = 2$$
$$\vdots$$
$$x[n] = n\,u[n]$$

$$X(z) = \frac{z}{z-1} \sum_{n=-\infty}^{\infty} u[m-1]\, z^{-n}$$

$$= \frac{z}{z-1}\left(\sum_{n=1}^{\infty} z^{-n}\right) = \frac{z}{z-1}\,\frac{1}{z}\left(\sum_{n=0}^{\infty} z^{-n}\right)$$

$$= \frac{z}{z-1}\,\frac{1}{z}\,\frac{z}{z-1} = \frac{z}{(z-1)^2} \qquad |z| > 1$$



one zero, two poles

Another one

$$y[n] = \sum_{m=-\infty}^{n} m\,u[m]$$

$$y[0] = 0$$

$$y[1] = 0 + 1 = 1$$

$$y[2] = 0 + 1 + 2 = 3$$

$$y[3] = 0 + 1 + 2 + 3 = 6$$

$$y[n] = \frac{n(n+1)}{2}\,u[n]$$

Think of accumulator function

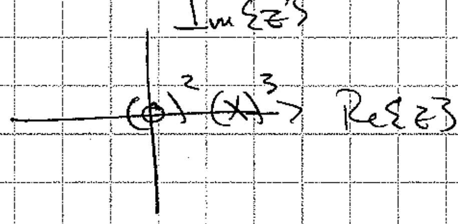
$$Y(z) = \frac{z}{z-1}\left(\sum_{n=-\infty}^{\infty} n\,u[n]\,z^{-n}\right)$$

$$= \frac{z}{z-1}\left(\sum_{n=0}^{\infty} n\,z^{-n}\right)$$

$$\uparrow$$

$$\frac{z}{(z-1)^2}$$

$$Y(z) = \frac{z^2}{(z-1)^3} \qquad |z| > 1$$



Differentiation

$$y[n] = x[n] - x[n-1]$$

$$Y(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} - \sum_{n=-\infty}^{\infty} x[n-1] z^{-n}$$

$$= X(z) - \sum_{n=-\infty}^{\infty} x[n] z^{-(n+1)}$$

$$= X(z) - \frac{1}{z} X(z)$$

$$= \left(1 - \frac{1}{z}\right) X(z)$$

Connection with accumulator?

$$y_a[n] = \sum_{m=-\infty}^{n} x[n] \longrightarrow Y_a(z) = \frac{z}{z-1} X(z)$$

$$y_d[n] = x[n] - x[n-1] \longrightarrow Y_d(z) = \frac{z-1}{z} X(z)$$

With this kind of definition, the two operators correspond to inverse functions.

Cool!

Example

$$x[n] = n\left(\tfrac{1}{2}\right)^n u[n]$$

$$X(z) = \sum_{n=-\infty}^{\infty} n\left(\tfrac{1}{2}\right)^n u[n]\, z^{-n}$$

Thinking.. we can sum the geometric part

$$\sum_{n=-\infty}^{\infty} \left(\tfrac{1}{2}\right)^n u[n]\, z^{-n} = \sum_{n=0}^{\infty} \left(\tfrac{1}{2}\right)^n z^{-n} = \sum_{n=0}^{\infty} \frac{1}{(2z)^n}$$

$$= \frac{1}{1 - 1/2z} = \frac{z}{z - \frac{1}{2}} \qquad \left|\tfrac{1}{2z}\right| < 1 \quad \text{or } |z| > \tfrac{1}{2}$$

But with the ~~n~~ n inside, it becomes more difficult! What we need is a trick..

In the case of the Laplace transform, we could take a derivative.

$$-\frac{d}{ds}\int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt = \int_{-\infty}^{\infty} t\, x(t)\, e^{-st}\, dt$$

Can we do something similar for the Z-transform?

$$-\frac{d}{dz}\sum_{n=-\infty}^{\infty} f[n]\, z^{-n} = \sum_{n=-\infty}^{\infty} n\, f[n]\, z^{-(n+1)}$$

$$= \frac{1}{z}\sum_{n=-\infty}^{\infty} n\, f[n]\, z^{-n}$$

OK, we get

$$\sum_{n=-\infty}^{\infty} n\, f[n]\, z^{-n} = -z\frac{d}{dz}\sum_{n=-\infty}^{\infty} f[n]\, z^{-n}$$

$$= -z\frac{d}{dz}F(z)$$

Looks like it works OK

So, $X(z) = \sum_{n=-\infty}^{\infty} n\left(\frac{1}{2}\right)^n u[n]\, z^{-n}$

$$= -z\frac{d}{dz}\underbrace{\sum_{n=-\infty}^{\infty}\left(\frac{1}{2}\right)^n u[n]\, z^{-n}}_{\frac{z}{z-1/2}}$$

$$= -z\frac{d}{dz}\frac{z}{z-1/2}$$

$$= -z\left(\frac{1}{z-1/2} - \frac{z}{(z-1/2)^2}\right)$$

$$= -z\left[\frac{z-1/2}{(z-1/2)^2} - \frac{z}{(z-1/2)^2}\right] = \frac{1}{2}\,\frac{z}{(z-1/2)^2}$$

See whether trick works on ramp...

$$x[n] = n\,u[n]$$

$$X(z) = \sum_{n=-\infty}^{\infty} n\,u[n]\,z^{-n}$$

$$= -z \frac{d}{dz} \sum_{n=-\infty}^{\infty} u[n]\,z^{-n}$$

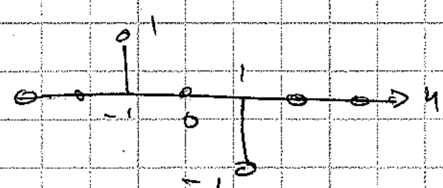
$$= -z \frac{d}{dz}\left(\frac{z}{z-1}\right) \qquad |z| > 1$$

$$= -z \left[\frac{1}{z-1} - \frac{z}{(z-1)^2}\right]$$

$$= -z \left[\frac{z-1}{(z-1)^2} - \frac{z}{(z-1)^2}\right]$$

$$= \frac{z}{(z-1)^2} \Longleftarrow$$ same answer as we got last time! Cool, it works!

Another example

$x[n] =$ 

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\,z^{-n} = z - \frac{1}{z} \qquad \text{what is ROC?}$$

z-transform converges for $|z| > 0$, signal is of finite duration.

Example: find $x[n]$ for

$$X(z) = \frac{1}{z^2}\frac{1}{(z-1)^2} \qquad \text{Assume } |z|>1$$

This problem seems like examples we did before. Recall that

$$\text{Ramp} = u[n]\,n \longleftrightarrow \frac{z}{(z-1)^2}$$

Ok, so

$$X(z) = \frac{1}{z^2}\frac{1}{(z-1)^2} = \frac{1}{z^3}\underbrace{\left[\frac{z}{(z-1)^2}\right]}_{\text{Ramp}}$$

~~Think~~ Think we have a shift...

$$x[n] = u[n-3](n-3)$$

Can we verify?

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=-\infty}^{\infty} u[n-3](n-3)z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} u[n]\,n\,z^{-(n+3)} = \frac{1}{z^3}\underbrace{\sum_{n=-\infty}^{\infty} u[n]\,n\,z^{-n}}_{\frac{z}{(z-1)^2}}$$

Example

$$x[n] = \cos\left(\frac{n\pi}{2}\right) u[n]$$

$$X(z) = \sum_{n=-\infty}^{\infty} \cos\left(\frac{n\pi}{2}\right) u[n]\, z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{2}\left[e^{j\frac{\pi n}{2}} + e^{-j\frac{\pi n}{2}} \right] z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{2}\left(\left(\frac{e^{j\frac{\pi}{2}}}{z}\right)^{n} + \left(\frac{e^{-j\frac{\pi}{2}}}{z}\right)^{n} \right) \qquad e^{j\frac{\pi}{2}} = j$$

$$e^{-j\frac{\pi}{2}} = -j$$

$$= \sum_{n=0}^{\infty} \frac{1}{2}\left[\left(\frac{j}{z}\right)^{n} + \left(\frac{-j}{z}\right)^{n} \right]$$

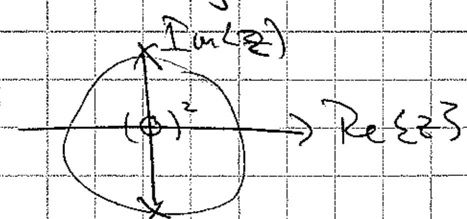
$$= \frac{1}{2}\left[\frac{1}{1-(j/z)} + \frac{1}{1-(-j/z)} \right] \qquad |z| > 1$$

$$= \frac{1}{2}\left[\frac{z}{z-j} + \frac{z}{z+j} \right]$$

$$= \frac{1}{2}\left[\frac{z(z+j) + z(z-j)}{z^2+1} \right]$$

$$\boxed{X(z) = \frac{z^2}{z^2+1}}$$

interesting...



$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

let

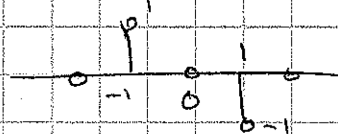
$$x[n] = -x[-n]$$

$$X(z) = -\sum_{n=-\infty}^{\infty} x[-n]\, z^{-n}$$

$$= -\sum_{n=-\infty}^{\infty} x[n]\, z^{n}$$

$$= -X\left(\tfrac{1}{z}\right)$$

Example:

$x[n] =$ 

$$X(z) = z - \tfrac{1}{z}$$

$$-X\left(\tfrac{1}{z}\right) = z - \tfrac{1}{z}$$ ok it works