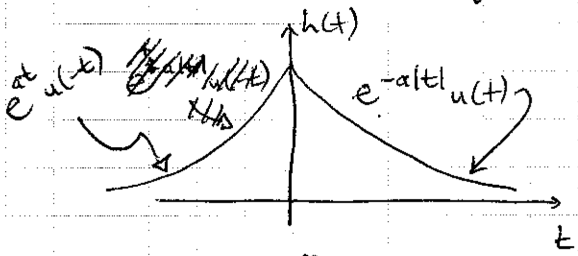
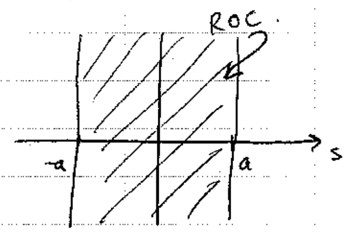
6.003 REC., SECTIONS 3 & 4.
Feb 24, 2010.

Today:

(1) Simple example to review key issues in laplace & z-transforms.
(2) Second-order CT LCR circuits with laplace transforms.
(3) Second-order DT block with z-transforms.

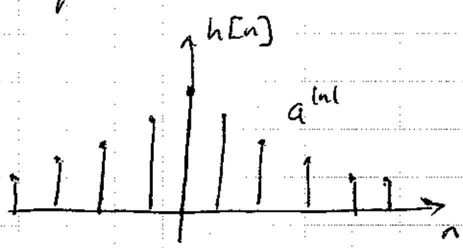
① Review: suppose we are given a CT impulse resp.:

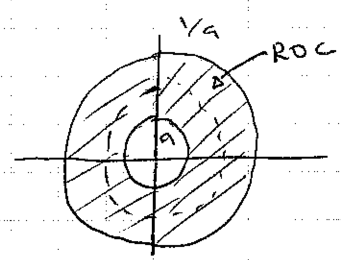




$$H(s) = \int_{-\infty}^{\infty} h(t) e^{-st}\,dt = \frac{1}{a-s} + \frac{1}{a+s} = \frac{2a}{a^2-s^2}$$

The equivalent DT IR is

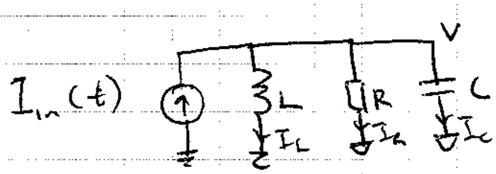




$$H(z) = \sum_{n=-\infty}^{+\infty} h[n] z^{-n} = \frac{az}{1-az} + \frac{1}{1-az^{-1}} = \frac{1-a^2}{(1-az)(1-az^{-1})}$$

In the laplace domain, note that intergration → 1/s, and differentiation → s.

②



$$I_L = \frac{1}{L}\int v\,dt \Leftrightarrow I_L(s) = \frac{V(s)}{sL} \Rightarrow Z_L(s) = \frac{V_L(s)}{I_L(s)} = sL.$$

$$I_R = V/R \Leftrightarrow I_R(s) = V_R(s)/R \Rightarrow Z_R(s) = R.$$

$$I_C = C\frac{dv}{dt} \Leftrightarrow I_C(s) = sCV \Rightarrow Z_C(s) = \frac{V_C(s)}{I_C(s)} = \frac{1}{sC}$$

L's impedance is sL, admittance is $1/sL$.
R's impedance is R, conductance is $G = 1/R$.
C's impedance is $1/sC$, admittance is sC.

$$\frac{I_L(s)}{I_{in}(s)} = \left(\frac{1}{\frac{1}{sL} + sC + 1/R}\right)\frac{1}{sL} = \frac{1}{LCs^2 + \frac{L}{R}s + 1}.$$

Similarly,

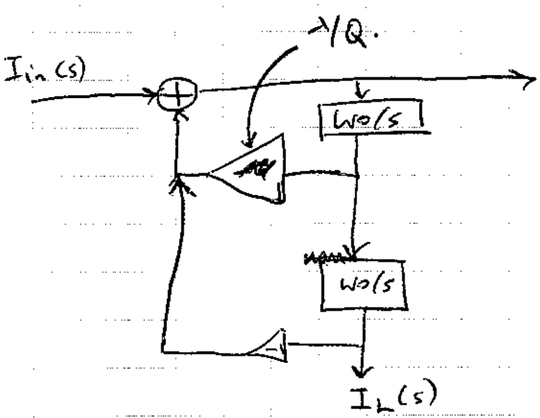
$$\frac{I_R(s)}{I_{in}(s)} = \frac{L/R\, s}{LCs^2 + L/R\, s + 1}$$

and

$$\frac{I_C(s)}{I_{in}(s)} = \frac{LCs^2}{LCs^2 + L/R\, s + 1}$$

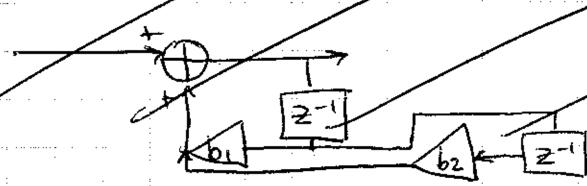
Define $LC \triangleq 1/\omega_0^2$, $Q \triangleq \frac{R}{\sqrt{L/C}}$, which generates

$$\frac{I_L(s)}{I_{in}(s)} = \frac{\omega_0^2}{s^2 + \frac{s\omega_0}{Q} + \omega_0^2}, \quad \frac{I_R(s)}{I_{in}(s)} = \frac{s/\omega_0 Q}{\frac{s^2}{\omega_0^2} + \frac{s}{\omega_0 Q} + 1}, \quad \frac{I_C(s)}{I_{in}(s)} = \frac{s^2/\omega_0^2}{s^2/\omega_0^2 + s/\omega_0 Q + 1}.$$

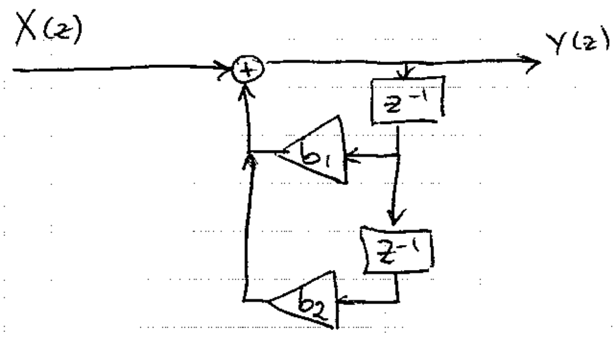


, $\frac{I_L(s)}{I_{in}(s)} = \frac{\omega_0^2/s^2}{(1 + \frac{\omega_0}{sQ} + \omega_0^2/s^2)}$ which is identical to what we derived by the impedance method.

~~(3) Canonical second order - DT system is represented by,~~



(3) A canonical second-order DT system is represented by



$$\frac{Y(z)}{X(z)} = \frac{1}{1 - b_1 z^{-1} - b_2 z^{-2}}$$

$$r_1 r_2 \triangleq -b_2$$
$$r_1 + r_2 \triangleq b_1$$

Suppose $r_1 = re^{+j\theta}$, $r_2 = re^{-j\theta}$

Note, roots are complex conjugates.

$$\frac{Y(z)}{X(z)} = \frac{1}{(1-r_1 z^{-1})(1-r_2 z^{-1})}$$

The inverse z-transform is then $\dfrac{r_1 \; r_1^n u[n]}{r_1 - r_2} + \dfrac{r_2 \; r_2^n u[n]}{r_1 - r_2}$

$$= \frac{r^{n+1} e^{j\theta(n+1)}}{re^{j\theta} - re^{-j\theta}} - \frac{r^{n+1} e^{-j\theta(n+1)}}{re^{j\theta} - re^{-j\theta}} = \frac{r^n \sin[(n+1)\theta]}{\sin\theta}.$$

Assume $r < 1$, so the impulse response is