6.003 R

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Z - Transforms, more examples

$$x[n] = \cos\left(\frac{\pi n}{2}\right) u[n]$$

$$X(z) = \sum_{n=-\infty}^{\infty} \cos\left(\frac{\pi n}{2}\right) u[n] z^{-n}$$

$$= \sum_{n=0}^{\infty} \cos\left(\frac{\pi n}{2}\right) z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{2}\left[e^{\frac{j\pi n}{2}} + e^{-j\frac{\pi n}{2}} \right] z^{-n}$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{e^{j\pi/2}}{z}\right)^n + \left(\frac{e^{-j\pi/2}}{z}\right)^n$$

Thinking...

$$e^{j\pi/2} = j$$

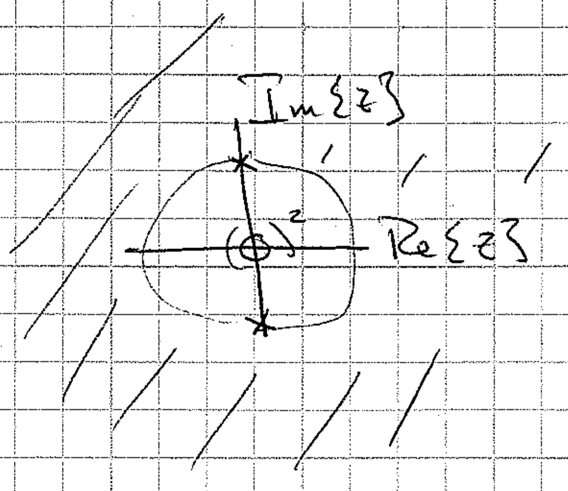
$$e^{-j\pi/2} = -j$$

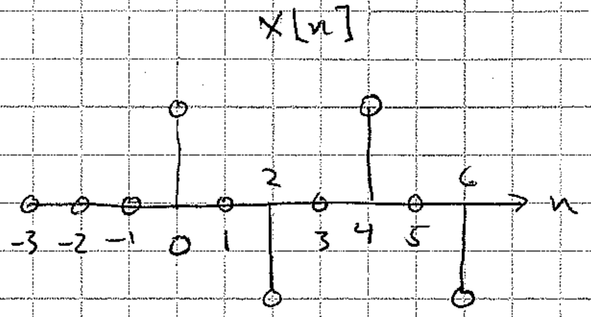
$$X(z) = \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{j}{z}\right)^n + \left(\frac{-j}{z}\right)^n$$

$$= \frac{1}{2}\left[\frac{1}{1 - j/z} + \frac{1}{1 + j/z} \right] \qquad |j/z| < 1 \qquad |z| > 1$$

$$= \frac{1}{2}\left[\cancel{1+j/z} \frac{z}{z-j} + \frac{z}{z+j} \right]$$

$$= \frac{1}{2}\left[\frac{z(z+j) + z(z-j)}{z^2+1} \right] = \frac{z^2}{z^2+1}$$



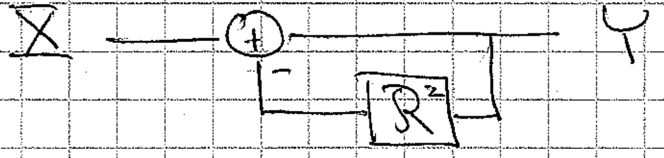


More thinking...

suppose $H(z) = \frac{z^2}{z^2+1}$

What does the system look like?

$$H(z) = \frac{1}{1+\left(\frac{1}{z^2}\right)} \longleftrightarrow H = \frac{1}{1+\mathcal{R}^2}$$



$$Y = X - \mathcal{R}^2 Y$$

$$(1+\mathcal{R}^2)Y = X$$

$$\frac{Y}{X} = \frac{1}{1+\mathcal{R}^2} = 1 - \mathcal{R}^2 + \mathcal{R}^4 - \mathcal{R}^6 + \cdots$$

<u>ok</u> makes sense

Example

$$x[n] = u[n]\left(\frac{1}{2}\right)^n \sin\left(\frac{n\pi}{3}\right)$$

$$X(z) = \sum_{n=-\infty}^{\infty} u[n]\left(\frac{1}{2}\right)^n \sin\left(\frac{n\pi}{3}\right) z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{(2z)^n} \sin\left(\frac{n\pi}{3}\right)$$

$$= \frac{1}{2j} \sum_{n=0}^{\infty} \frac{1}{(2z)^n}\left[e^{j\frac{n\pi}{3}} - e^{-j\frac{n\pi}{3}}\right]$$

$$= \frac{1}{2j} \sum_{n=0}^{\infty} \left(\frac{e^{j\pi/3}}{2z}\right)^n - \left(\frac{e^{-j\pi/3}}{2z}\right)^n$$

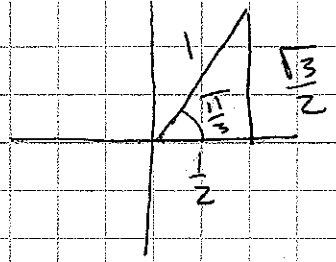
$$= \frac{1}{2j}\left\{\frac{1}{1-\frac{e^{j\pi/3}}{2z}} - \frac{1}{1-\frac{e^{-j\pi/3}}{2z}}\right\}$$

$$= \frac{1}{2j}\left\{\frac{z}{z - e^{j\pi/3}/2} - \frac{z}{z - \frac{e^{-j\pi/3}}{2}}\right\}$$

$$= \frac{1}{2j}\left\{\frac{z^2 - ze^{-j\pi/3}/2 - z^2 + ze^{j\pi/3}/2}{z^2 - \frac{z}{2}\left(e^{j\pi/3} + e^{-j\pi/3}\right) + \frac{1}{4}}\right\}$$

$$= \frac{1}{4j}\left\{\frac{z\left(e^{j\pi/3} - e^{-j\pi/3}\right)}{z^2 - \frac{z}{2}\left(e^{j\pi/3} + e^{-j\pi/3}\right) + \frac{1}{4}}\right\}$$

$$= \frac{1}{2}\left\{\frac{z\sin(\pi/3)}{z^2 - z\cos(\pi/3) + \frac{1}{4}}\right\}$$



$$\cos\frac{\pi}{3} = \frac{1}{2}$$

$$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$$X(z) = \frac{1}{2}\,\frac{z\,\sqrt{3}/2}{z^2 - \frac{z}{2} + \frac{1}{4}}$$

$$= \frac{\sqrt{3}}{4}\left(\frac{z}{z^2 - \frac{z}{2} + \frac{1}{4}}\right)$$

OK. so, where are the poles?

$$z^2 - \frac{z}{2} + \frac{1}{4} = 0$$

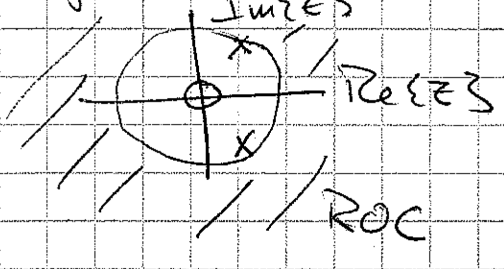
$$z = \frac{\frac{1}{2} \pm \sqrt{\frac{1}{4} - 1}}{2}$$

$$= \frac{1}{4} \pm j\,\frac{\sqrt{\frac{3}{4}}}{2}$$

$$= \frac{1}{4} \pm j\,\frac{\sqrt{3}}{4}$$

More thinking, looking for $\frac{e^{\pm j\pi/3}}{2} = \frac{\frac{1}{2} \pm j\sqrt{3}/2}{2}$

OK. so poles are in the right place



Can we do the inverse transform?

$$X(z) = \frac{\sqrt{3}}{4}\left(\frac{z}{z^2 - \frac{z}{2} + \frac{1}{4}}\right)$$

$$= \frac{Az}{z - \frac{e^{j\pi/3}}{2}} + \frac{Bz}{z - \frac{e^{j\pi/3}}{2}}$$

$$= \frac{Az^2 - Az\frac{e^{-j\pi/3}}{2} + Bz^2 - Bz\frac{e^{+j\pi/3}}{2}}{z^2 - z\left(\frac{e^{j\pi/3} + e^{-j\pi/3}}{2}\right) + \frac{1}{4}}$$

$$= \frac{(A+B)z^2 - \frac{z}{2}\left(Ae^{-j\pi/3}\cancel{z} + Be^{j\pi/3}\right)}{z^2 - \frac{z}{2} + \frac{1}{4}}$$

Need $A + B = 0$, so ~~$B =$~~ $A = -B$

$$= -\frac{B}{2}\,\frac{z\left(e^{j\pi/3} - e^{-j\pi/3}\right)}{z^2 - \frac{z}{2} + \frac{1}{4}}$$

$$= -jB\,\frac{z\sin(\pi/3)}{z^2 - \frac{z}{2} + \frac{1}{4}} \Rightarrow -jB\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4}$$

$$B = -\frac{1}{2j}$$

$$A = +\frac{1}{2j}$$

$$X(z) = +\frac{1}{2j}\,\frac{z}{z - \frac{e^{j\pi/3}}{2}} - \frac{1}{2j}\,\frac{z}{z e^{-j\pi/3}}$$

$$x[n] = \frac{1}{2j}\,u[n]\left(\frac{1}{2}\right)^n e^{j\frac{\pi n}{3}}$$

$$- \frac{1}{2j}\,u[n]\left(\frac{1}{2}\right)^n e^{-j\pi n/3}$$

~~$= \left[u[n]\left(\frac{1}{2}\right)^n e^{j\frac{\pi n}{3}} -\right.$~~

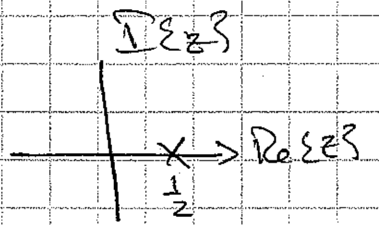
$$= u[n]\left(\frac{1}{2}\right)^n \left[\frac{e^{j\frac{\pi n}{3}} - e^{-j\frac{\pi n}{3}}}{2j}\right]$$

$$\boxed{x[n] = u[n]\left(\frac{1}{2}\right)^n \sin\left(\frac{\pi n}{3}\right)}$$

OK - it works!

Example

Start with pole-zero plot, get impulse function



ROC $|z| > \frac{1}{2}$

$$H(z) = \frac{1}{z - \frac{1}{2}}$$

$$Y(z) = \frac{1}{z - 1/2} X(z)$$

$$\left(z - \tfrac{1}{2}\right) Y(z) = X(z)$$

$$y[n+1] - \tfrac{1}{2} y[n] = x[n]$$

Check -

$$\sum_{n=-\infty}^{\infty} y[n+1] z^{-n} - \frac{1}{2} \sum_{n=-\infty}^{\infty} y[n] z^{-n} = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$\downarrow$$

$$\sum_{n=-\infty}^{\infty} y[n] z^{-(n-1)} - \quad '' \qquad\qquad ''$$

$$\downarrow$$

$$z\,Y(z) - \tfrac{1}{2} Y(z) = X(z)$$

$$Y(z) = \frac{1}{z - \frac{1}{2}} X(z)$$

n	x[n]	y[n]
-1	0	0
0	1	0
1	0	1
2	0	$\frac{1}{2}$
3	0	$\frac{1}{4}$
⋮	⋮	⋮

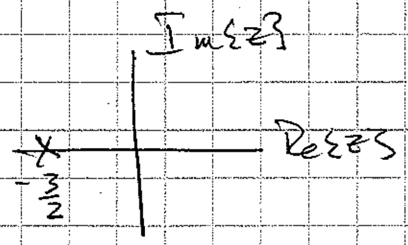
$$y[n+1] = \frac{1}{2}y[n] + x[n]$$

$$y[n] = \frac{1}{2}y[n-1] + x[n-1]$$

$$y[n] = u[n-1]\left(\frac{1}{2}\right)^{(n-1)}$$

What does it mean? pole seems to correspond to the decay part of the impulse Response.

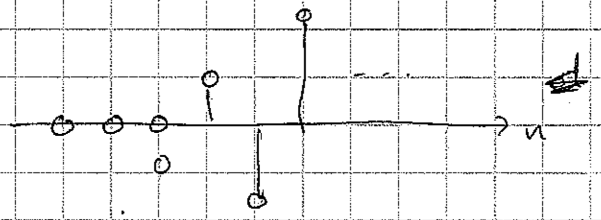
Example



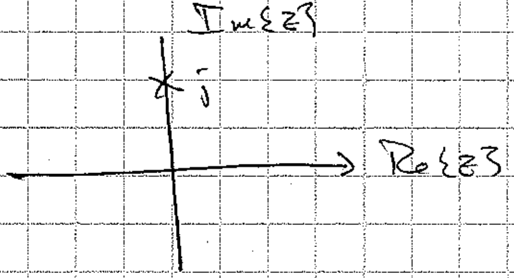
ROC $|z| > \frac{3}{2}$

Expect very similar...

$$h[n] = u[n-1]\left(\frac{-3}{2}\right)^{(n-1)}$$



Example



$$h[n] = u[n-1]\,(j)^{(n-1)}$$

$$= u[n-1]\left[e^{\frac{j\pi}{2}}\right]^{n-1}$$

$$= u[n-1]\,e^{-\frac{j\pi}{2}}\,e^{\frac{j\pi n}{2}}$$

$$= -j\,u[n-1]\left(\cos\left(\frac{n\pi}{2}\right) + j\sin\left(\frac{n\pi}{2}\right)\right)$$

Example

$$x[n] = u[n]\, n \sin(\alpha n)$$

$$X(z) = \sum_{n=-\infty}^{\infty} u[n]\, n \sin(\alpha n)\, z^{-n}$$

$$= \sum_{n=0}^{\infty} n \sin(\alpha n)\, z^{-n}$$

$$= \sum_{n=0}^{\infty} n \left[\frac{e^{j\alpha n} - e^{-j\alpha n}}{2j}\right] z^{-n}$$

$$= \frac{1}{2j} \sum_{n=0}^{\infty} n \left[\left(\frac{e^{j\alpha}}{z}\right)^{n} - \left(\frac{e^{-j\alpha}}{z}\right)^{n}\right]$$

$$F(z) = \frac{1}{2j} \sum_{n=0}^{\infty} \left(\frac{e^{j\alpha}}{z}\right)^{n} - \left(\frac{e^{-j\alpha}}{z}\right)^{n}$$

$$f[n] = u[n] \sin(\alpha n)$$

$$F(z) = \frac{1}{2j} \left[\frac{1}{1 - e^{j\alpha}/z} - \frac{1}{1 - e^{-j\alpha}/z}\right]$$

$$= \frac{1}{2j} \left[\frac{z}{z - e^{j\alpha}} - \frac{z}{z - e^{-j\alpha}}\right]$$

$$= \frac{1}{2j} \left[\frac{z^2 - z e^{-j\alpha} - z^2 + z e^{j\alpha}}{z^2 - z(e^{j\alpha} + e^{-j\alpha}) + 1}\right]$$

$$= \frac{z \sin(\alpha)}{z^2 - 2z\cos(\alpha) + 1}$$

$$\overline{\underline{X}}(z) = -z \frac{d}{dz} F(z)$$

$$= -z \frac{d}{dz} \left[\frac{z \sin\alpha}{z^2 - 2z\cos\alpha + 1} \right]$$

$$= -z \left[\frac{\sin\alpha}{z^2 - 2z\cos\alpha + 1} - \frac{z \sin\alpha \left(2z - 2\cos\alpha \right)}{\left(z^2 - 2z\cos\alpha + 1 \right)^2} \right]$$

$$= -z \left[\frac{\sin\alpha \left(z^2 - 2z\cos\alpha + 1 \right) - 2z\sin\alpha \left(z - \cos\alpha \right)}{\left(z^2 - 2z\cos\alpha + 1 \right)^2} \right]$$

$$= -z \left[\frac{-z^2 \sin\alpha + \sin\alpha}{\left(z^2 - 2z\cos\alpha + 1 \right)^2} \right]$$

$$= \sin\alpha \left[\frac{z(z^2 - 1)}{\left(z^2 - 2z\cos\alpha + 1 \right)^2} \right]$$