6.003 Recitation, Sections 3 & 4, FEB. 26, 2010

Today: 1) Properties of Laplace & Z-transforms
2) Examples of applying these for Laplace transforms
3) " " " " Z- " .

(1)

only if the system starts at rest.

| | Laplace | | Z-transform | |
|---|---|---|---|---|
| Linearity | $ax(t) + by(t)$ | $\underbrace{aX(s) + bY(s)}$ | | $aX[n] + bY[n]$ |
| Differentiation | $dx/dt$ | $sX(s)$ | $x[n]-x[n-1]$ ⇒ | $(1-z^{-1})X(z)$ |
| Integration | $\int_{-\infty}^{t} x(t)dt$ | $X(s)/s$ | $\sum_{k=-\infty}^{k=n} x[k]$ ⇒ | $\left(\frac{1}{1-z^{-1}}\right)X(z)$ |
| Shift | $x(t-T)$ | $e^{-sT}X(s)$ | $x[n-N]$ | $z^{-N}X(z)$ |
| Multiply by $t$ | $tx(t)$ | $\frac{-dX(s)}{dt}$ | $nx[n]$ | $-z\,dX(z)/dz$ |
| Multiply by $e^{at}$ | $e^{at}x(t)$ | $X(s-a)$ | $a^n x[n]$ | $X(z/a)$ |
| Unit impulse | $\delta(t)$ | $1$ | $\delta[n]$ | $1$ |
| Unit step | $u(t)$ | $1/s$ | $u[n]$ | $\frac{1}{1-z^{-1}}$ |
| Exponential | $e^{-at}u(t)$ | $1/(s+a)$ | $a^n u[n]$ | $\frac{1}{1-az^{-1}}$ |
| Scaling | $x(at)$ | $\frac{1}{\lvert a\rvert}X(s/a)$ | $x[n]=0$ if $\frac{n}{k}\notin\mathbb{Z}$; $x[n]=x(n/k)$ if $n/k\in\mathbb{Z}$. | $X(z^k)$ |

(2) Eg 1) $\mathcal{L}\{t^n u(t)\}$

Take a simple example to start: $L\{tu(t)\} = -\frac{d}{ds}(1/s) = 1/s^2$.

$L\{t^2u(t)\} = 2/s^3$

$L\{t^3u(t)\} = 2\cdot3/s^4$

$\Rightarrow \mathcal{L}\{t^n u(t)\} = n!/s^{n+1}$.

Eg 2) $\mathcal{L}\{t^n e^{-at}u(t)\} \Rightarrow \frac{n!}{(s+a)^{n+1}}$.

Eg 3

$$\sin w_0 t\, u(t) = \frac{e^{jw_0 t} - e^{-jw_0 t}}{2j} u(t)$$

$$\mathcal{L}\{\sin w_0 t\, u(t)\} = \frac{1}{2j}\left\{\frac{1}{s-jw_0} + \frac{1}{s+jw_0}\right\} = \frac{1}{2j}\left(\frac{2jw_0}{s^2+w_0^2}\right) = \frac{w_0}{s^2+w_0^2}.$$

Eg 4

$$\cos(w_0 t)u(t) = \frac{e^{jw_0 t} + e^{-jw_0 t}}{2} = \frac{1}{2}\left\{\frac{1}{s-jw_0} + \frac{1}{s+jw_0}\right\} = \frac{s}{s^2+w_0^2}.$$

Eg 5

$$\mathcal{L}\{e^{-\sigma t}\sin(w_0 t+\phi)\} = \mathcal{L}\{e^{-\sigma t}[\sin w_0 t \cos\phi + \cos w_0 t \sin\phi]\}$$

$$= \cos\phi\left(\frac{w_0}{(s+\sigma)^2+w_0^2}\right) + \sin\phi\left(\frac{s+\sigma}{(s+\sigma)^2+w_0^2}\right)$$

3) Eg 1 $\quad Z\{nu[n]\} = -z\frac{d}{dz}\left(\frac{1}{1-z^{-1}}\right) = \frac{z^{-1}}{(1-z^{-1})^2}$

Eg 2 $\quad Z\{na^n u[n]\} = \dfrac{(z/a)^{-1}}{\left(1-(z/a)^{-1}\right)^2}$

Eg 3 $\quad Z\{\sin(\theta_0 n)u[n]\} = Z\left\{\dfrac{e^{j\theta_0 n} - e^{-j\theta_0 n}}{2j}u[n]\right\} = \dfrac{1}{2j}\left\{\dfrac{1}{1-e^{j\theta_0}z^{-1}} - \dfrac{1}{1-e^{-j\theta_0}z^{-1}}\right\}$

$$= \frac{\sin\theta_0 z^{-1}}{1-2\cos\theta_0 z^{-1} + z^{-2}}$$

Eg 4 $\quad Z\{r^n \sin(\theta_0 n)u[n]\} \Longrightarrow \dfrac{\sin\theta_0 z^{-1}}{1-2\cos\theta_0 z^{-1}+z^{-2}} \longrightarrow \dfrac{\sin\theta_0 r z^{-1}}{1-2\cos\theta_0 r z^{-1}+r^2 z^{-2}}.$