6.003R   March 5, 2010

Example: DT convolution   Peter Hagelstein

$$x[n] = \alpha^n u[n]$$

$$y[n] = \beta^n u[n]$$

$$(x * y)[n] = \sum_{k=-\infty}^{\infty} x[k]\, y[n-k]$$

$$= \sum_{k=-\infty}^{\infty} \alpha^k u[k]\, \beta^{(n-k)} u[n-k]$$

$$= \sum_{k=0}^{\infty} \alpha^k \beta^{(n-k)} u[n-k]$$

$$= \sum_{k=0}^{n} \alpha^k \beta^{(n-k)} \qquad \text{for } n \geq 0$$

$$= \beta^n \sum_{k=0}^{n} \left(\frac{\alpha}{\beta}\right)^k$$

Assume $\left(\frac{\alpha}{\beta}\right) < 1$ ...

(Get same answer if $\left(\frac{\beta}{\alpha}\right) < 1$)

$$= \beta^n \left(\sum_{k=0}^{\infty} \left(\frac{\alpha}{\beta}\right)^k - \sum_{k=n+1}^{\infty} \left(\frac{\alpha}{\beta}\right)^k \right)$$

$$= \beta^n \left(\frac{1}{1 - \alpha/\beta} - \cancel{\tfrac{\alpha}{\beta}} \sum_{k=0}^{\infty} \left(\frac{\alpha}{\beta}\right)^{n+1+k} \right)$$

$$= \beta^n \left(\frac{1}{1 - \alpha/\beta} - \left(\frac{\alpha}{\beta}\right)^{n+1} \frac{1}{1 - \alpha/\beta} \right)$$

$$= \beta^n \frac{1 - (\alpha/\beta)^{n+1}}{1 - (\alpha/\beta)} \Rightarrow \frac{\beta^{n+1} - \alpha^{n+1}}{\beta - \alpha} u[n]$$

Example CT convolution

$$x(t) = u(t)\, e^{-\alpha t}$$

$$y(t) = u(t)\, e^{-\beta t}$$

$$(x * y)(t) = \int_{-\infty}^{\infty} u(\tau) e^{-\alpha\tau}\, u(t-\tau) e^{-\beta(t-\tau)}\, d\tau$$

$$= \int_{0}^{\infty} e^{-\alpha\tau}\, u(t-\tau) e^{-\beta(t-\tau)}\, d\tau$$

$$= \begin{cases} 0 & \text{for } t<0 \\ \int_0^t e^{-\alpha\tau} e^{-\beta(t-\tau)}\, d\tau & \text{for } t \geq 0 \end{cases}$$

$$= u(t)\, e^{-\beta t} \int_0^t e^{-(\alpha-\beta)\tau}\, d\tau$$

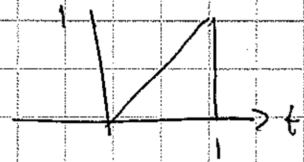
$$= u(t)\, e^{-\beta t} \left. \frac{e^{-(\alpha-\beta)\tau}}{-(\alpha-\beta)} \right|_0^t$$

$$= u(t)\, e^{-\beta t} \left\{ \frac{e^{-(\alpha-\beta)t}}{-(\alpha-\beta)} - \frac{1}{-(\alpha-\beta)} \right\}$$

$$= u(t)\, e^{-\beta t} \left\{ \frac{1 - e^{-(\alpha-\beta)t}}{\alpha-\beta} \right\}$$

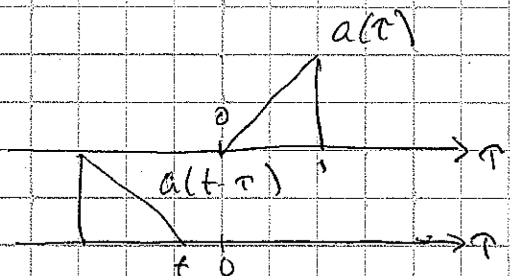
$$= u(t) \left(\frac{e^{-\beta t} - e^{-\alpha t}}{\alpha - \beta} \right)$$

Example

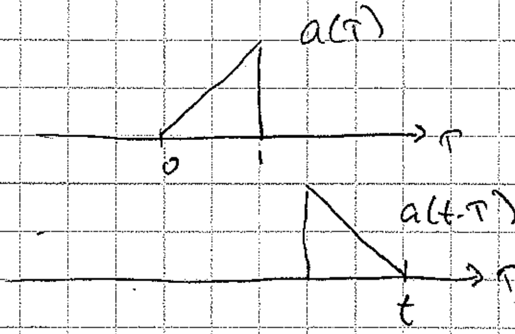
$a(t) =$ 

$$y(t) = (a * a)(t)$$

$$= \int_{-\infty}^{\infty} a(\tau)\, a(t-\tau)\, d\tau$$

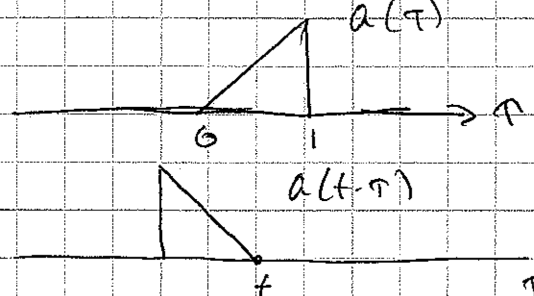


for $t < 0$, no overlap, so $y(t) = 0$ for $t < 0$



for $t > 2$, no overlap, so $y(t) = 0$ for $t > 2$

for $0 < t < 1$



$$y(t) = \int_0^t a(\tau)\, a(t-\tau)\, d\tau$$

$$= \int_0^t \tau\,(t-\tau)\, d\tau = \tau\frac{t^2}{2} - \frac{\tau^3}{3}\Big|_0^t = \frac{t^3}{2} - \frac{t^3}{3}$$

$$= \frac{t^3}{6}$$

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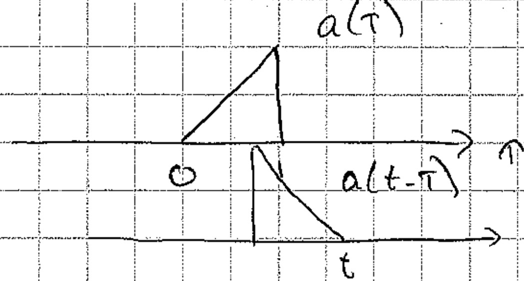
for $1 < t < 2$



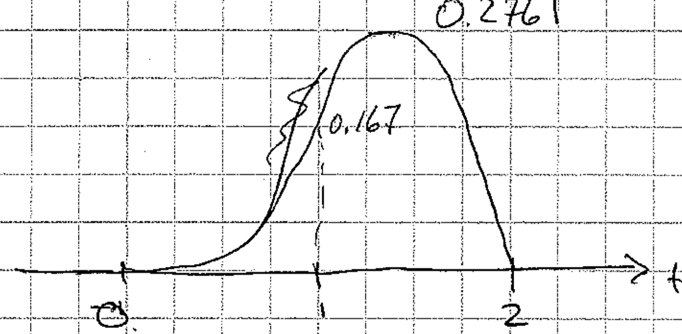
$$y(t) = \int_{t-1}^{1} a(\tau)\, a(t-\tau)\, d\tau$$

$$= \int_{t-1}^{1} \tau\,(t-\tau)\, d\tau = \left(\frac{t\tau^2}{2} - \frac{\tau^3}{3}\right)\Big|_{t-1}^{1}$$

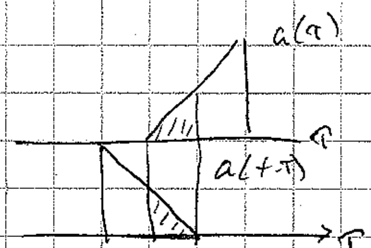
$$= \left(\frac{t}{2} - \frac{1}{3}\right) - \frac{t(t-1)^2}{2} + \frac{(t-1)^3}{3}$$

$$= -\frac{2}{3} + t - \frac{t^3}{6}$$

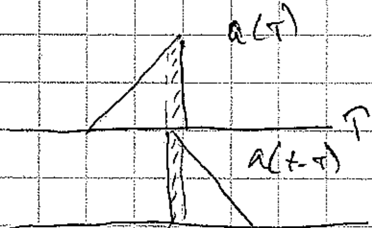
Put it together



Could we see this graphically?



for t near 0, probably could argue for $t^3/6$ behavior by staring at it...



For t near 2, know that it should start at 0, then increase linear ~~and~~ as t decreases

Anti causal (or acausal, noncausal) systems

$$X \longrightarrow \boxed{\mathcal{L}} \longrightarrow Y$$

$$y[n] = x[n+1]$$

the output predicts the next input

Z-transform:

$$\sum_{n=-\infty}^{\infty} y[n]\, z^{-n} = \sum_{n=-\infty}^{\infty} x[n+1]\, z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x[n]\, z^{-(n-1)}$$

$$Y(z) = z\, X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = z$$

Is it possible to make a system that approximates this model with causal elements?

How to proceed...

Try Taylor series

$$x[n+1] = x[n] + \left(\frac{dx}{dn}\right) + \frac{1}{2}\frac{d^2x}{dn^2} + \frac{1}{3!}\frac{d^3x}{dn^3} + \cdots$$

Use $\frac{dx}{dn} \approx \frac{x[n]-x[n-1]}{\Delta n = 1}$ for lowest order predictor

$$y[n] = x[n] + \left(x[n]-x[n-1]\right)$$

$$\boxed{y[n] = 2x[n] - x[n-1]}$$ lowest-order predictor

This one gets right answer for linear $x[n]$

let $x[n] = \alpha n$

$$y[n] = 2\alpha n - \alpha(n-1)$$

$$= \alpha n + \alpha = \alpha(n+1) = x[n+1]$$

Next-order, need better approximation

~~Require~~

$$y[n] = A\,x[n] + B\,x[n-1] + C\,x[n-2]$$

Want it to be OK for $x[n] = \alpha + \beta n + \gamma n^2$

For a constant

$$x[n+1] = \alpha = A\alpha + B\alpha + C\alpha$$

$$\Rightarrow \quad A + B + C = 1$$

Linear $\quad x[n+1] = \beta(n+1) = A\beta n + B\beta(n-1) + C\beta(n-2)$

$$1 = -B - 2C$$

Quadratic

$$x[n+1] = \gamma(n+1)^2 = A\gamma n^2 + B\gamma(n-1)^2 + C\gamma(n-2)^2$$

$$1 = B + 4C$$

Get

$$A = 3, \quad B = -3, \quad C = 1$$

$$y[n] = 3x[n] - 3x[n-1] + x[n-2]$$

Think about it ~~in~~ using Z-transform

$$Y(z) = \left(3 - \frac{3}{z} + \frac{1}{z^2}\right) X(z)$$

Slowly varying signals ~~lives~~ near $z=1$ in complex z-plane

$$H(z) = 3 - \frac{3}{z} + \frac{1}{z^2}$$

check near $z=1$

$$H(1) = 1$$

$$\frac{dH}{dz}(1) = \frac{3}{z^2} - \frac{2}{z^3}\bigg|_{z=1} = 1$$

$$\frac{d^2H}{dz^2}(1) = -\frac{6}{z^3} + \frac{6}{z^4}\bigg|_{z=1} = 0$$

Compare with ~~a~~ predictor

$$H_p(z) = z$$

$$H_p(1) = 1$$

$$\frac{dH_p}{dz}(1) = 1$$

$$\frac{d^2H_p}{dz^2}(1) = 0$$

So approximate predictor has $H(z)$ that looks like z near $z=1$

Example : solution of relaxation equation

$$\frac{dy}{dt} + \frac{y}{\tau} = x \quad \longleftarrow \text{Relaxation model}$$

Solve using 18.03 type of method

$$e^{-t/\tau}\frac{d}{dt}\left(e^{t/\tau} y\right) = x$$

$$\frac{d}{dt}\left(e^{t/\tau} y\right) = e^{t/\tau} x$$

$$\int_{-\infty}^{t} \frac{d}{dt'}\left(e^{t'/\tau} y\right) dt' = \int_{-\infty}^{t} e^{t'/\tau} x(t')\,dt'$$

$$e^{t'/\tau} y(t')\Big|_{-\infty}^{t} = e^{t/\tau} y(t) - \underbrace{e^{-\infty/\tau} y(-\infty)}_{\text{assume} = 0} = \int_{-\infty}^{t} e^{t'/\tau} x(t')\,dt'$$

$$y(t) = e^{-t/\tau} \int_{-\infty}^{t} e^{t'/\tau} x(t')\,dt'$$

$$= \int_{-\infty}^{t} x(t')\, e^{-(t-t')/\tau}\, dt'$$

$$= \int_{-\infty}^{\infty} x(t')\, e^{-(t-t')/\tau}\, u(t-t')\, dt'$$

This is in the form of a convolution

let $h(t) = e^{-t/\tau}\, u(t)$

$$y(t) = \int_{-\infty}^{\infty} x(t')\, h(t-t')\, dt'$$

$$= (x * h)(t)$$

Double check to see $h(t)$ is ~~impl~~ impulse response

$$\frac{dy}{dt} + \frac{y}{\tau} = \delta(t)$$

$$\left(s + \frac{1}{\tau}\right) Y(s) = 1$$

$$Y(s) = \frac{1}{s + 1/\tau}$$

$$y(t) = u(t)\, e^{-t/\tau} = h(t)$$

OK, it works.