6.003R Peter Hagelstein March 10, 2010

"A solution in search of a problem"

Second-order oscillatory systems with loss are sometimes hard for students to digest. Usually the example starts with a physical system, then a lossy oscillatory model is put forth as an approximation, and finally the mathematical machinary gets pulled out to analyze it.

However, lots of times what the Prof is trying to draw attention to is that the lossy oscillatory problem is useful. The example is just an example.

So, here we will begin with the solution, and work backwards...

Simple example

$$x \longrightarrow \boxed{h} \longrightarrow y = x * h$$

Try a simple $h(t)$:

$$h(t) = e^{-t/\tau}\, u(t)$$

OK ... so what is the Laplace transform

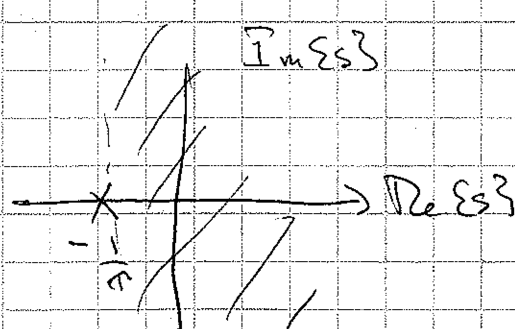
$$H(s) = \int_{-\infty}^{\infty} h(t)\, e^{-st}\, dt$$

$$= \int_{-\infty}^{\infty} e^{-t/\tau}\, u(t)\, e^{-st}\, dt$$

$$= \int_{0}^{\infty} e^{-(s+1/\tau)t}\, dt$$

$$= \frac{1}{s + 1/\tau} \qquad \mathrm{Re}\{s + \tfrac{1}{\tau}\} > 0$$

or $\mathrm{Re}\{s\} > -\frac{1}{\tau}$



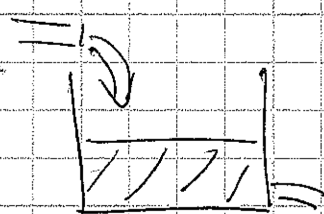
OK... so what was the differential eqn?

$$Y(s) = \frac{1}{s+\frac{1}{\tau}} X(s)$$

$$\left(s+\frac{1}{\tau}\right) Y(s) = X(s)$$

$$\left(\frac{d}{dt}+\frac{1}{\tau}\right) y(t) = x(t)$$

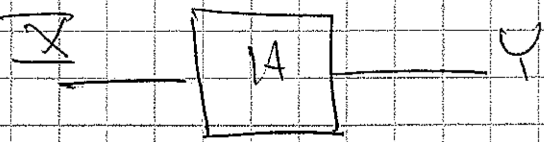
Hmmm... can we think of a model that works this way?



How about a leaky tank of water...

This logic might be behind an example taught at another school (we don't work this way at MIT...)

Another example



$$h(t) = e^{j\omega_0 t} e^{-t/\tau} u(t)$$

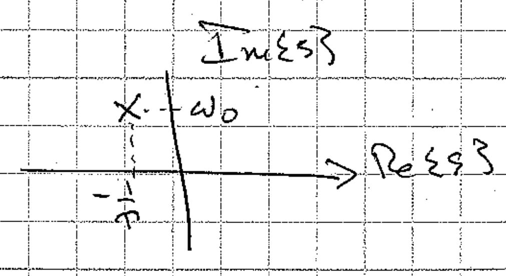
OK, this is a math example, since we are going to be hard pressed to come up with a physical example...

So, what is $H(s)$?

$$H(s) = \int_{-\infty}^{\infty} e^{j\omega_0 t} e^{-t/\tau} u(t) e^{-st} dt$$

$$= \int_{0}^{\infty} e^{-(s + \frac{1}{\tau} - j\omega_0)t} dt$$

$$= \frac{1}{s + \frac{1}{\tau} - j\omega_0} \qquad Re\{s\} > -\frac{1}{\tau}$$



get a single pole with complex s.

Fourier Transform?

$$\mathcal{F}(h(t)) = H(j\omega)$$

We will probably do Fourier transforms in their own right later on...

Magnitude and phase of $H(j\omega)$?

$$H(j\omega) = |H(j\omega)|\, e^{j\angle H(j\omega)}$$

$$H(j\omega) = \frac{1}{j(\omega-\omega_0)+\frac{1}{\tau}}$$

$$|H(j\omega)| = \sqrt{\frac{1}{(\omega-\omega_0)^2 + 1/\tau^2}}$$

Angle is harder to get, but we can...

$$H(j\omega) = \frac{-j(\omega-\omega_0)+1/\tau}{-j(\omega-\omega_0)+1/\tau} \cdot \frac{1}{j(\omega-\omega_0)+1/\tau}$$

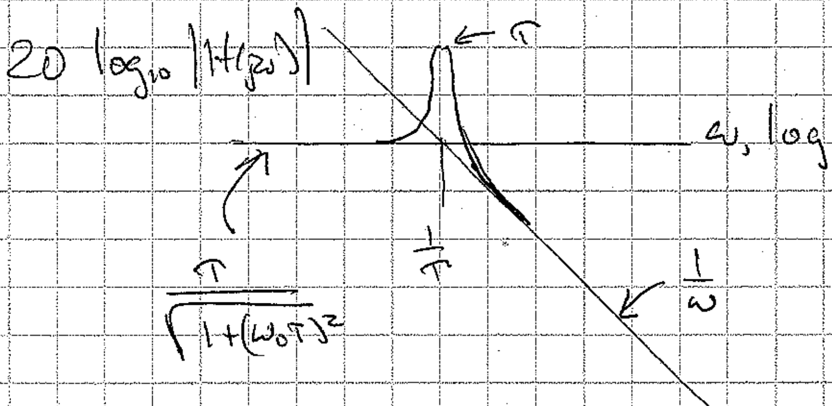
$$= \frac{-j(\omega-\omega_0)+1/\tau}{(\omega-\omega_0)^2 + 1/\tau^2}$$

$$\sin\left(\angle H(j\omega)\right) = -\frac{(\omega-\omega_0)}{\sqrt{(\omega-\omega_0)^2 + 1/\tau^2}}$$

$$\cos\left(\angle H(j\omega)\right) = \frac{1/\tau}{\sqrt{(\omega-\omega_0)^2 + 1/\tau^2}}$$

$$\tan\left(\angle H(j\omega)\right) = -\frac{(\omega-\omega_0)}{1/\tau} = -(\omega-\omega_0)\tau$$

What about Bode plot?



$\omega \to 0$

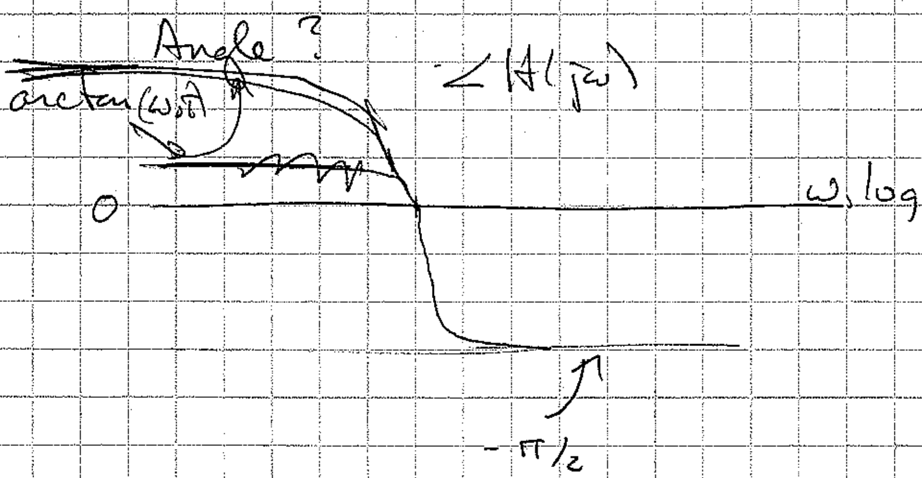
$$|H(j\omega)| = \frac{1}{\sqrt{\omega_0^2 + \frac{1}{\tau^2}}}$$

$\omega \to \omega_0$

$$|H(j\omega)| = \tau$$

$\omega \to \infty$

$$|H(j\omega)| = \frac{1}{|j\omega|} = \frac{1}{\omega}$$



$\omega \to 0$

$$\tan\left[\angle H(j\omega)\right] \to \omega_0\tau$$

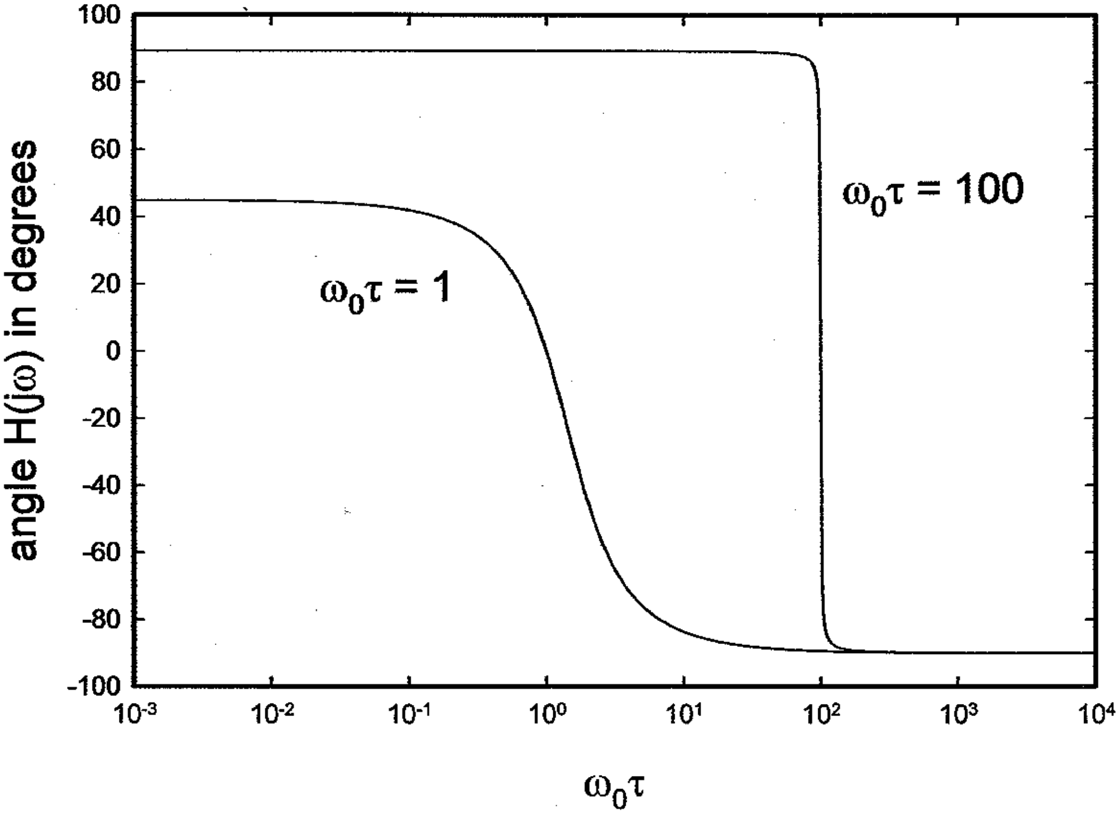
$\omega = \omega_0$

$$\angle H(j\omega) = 0$$

$\omega \to \infty$

$$\angle H(j\omega) = -\frac{\pi}{2}$$

large $\omega_0\tau$ $\quad \arctan(\omega_0\tau) \to \frac{\pi}{2}$



100
80
60
40
20
0
-20
-40
-60
-80
-100
angle H(jω) in degrees
ω0τ = 100
ω0τ = 1
10-3
10-2
10-1
100
101
102
103
104
ω0τ

Quality factor

$$Q = \frac{\omega_0 \tau}{2}$$

How to interpret?

Think about $|h(t)|^2 = e^{-2t/\tau} = e^{-\gamma t}$

$$Q = \frac{\omega_0}{\gamma} = \frac{\text{Radial frequency}}{\text{"energy" loss rate}}$$

Another example

$$X \longrightarrow \boxed{H(s)} \longrightarrow Y$$

$$h(t) = \sin(\omega_0 t)\, e^{-t/\tau}\, u(t)$$

What is $H(s)$?

$$H(s) = \int_{-\infty}^{\infty} h(t)\, e^{-st}\, dt$$

$$= \int_{-\infty}^{\infty} \sin(\omega_0 t)\, e^{-t/\tau}\, u(t)\, e^{-st}\, dt$$

$$= \int_{0}^{\infty} e^{-(s+\frac{1}{\tau})t} \left(\frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j} \right) dt$$

$$= \frac{1}{2j} \left[\frac{1}{s + \frac{1}{\tau} - j\omega_0} - \frac{1}{s + \frac{1}{\tau} + j\omega_0} \right] \qquad \mathrm{Re}\{s\} > -\frac{1}{\tau}$$

$$= \frac{\omega_0}{\left(s + \frac{1}{\tau}\right)^2 + \omega_0^2}$$

Differential equation

$$\left[\left(s+\frac{1}{\tau}\right)^2+\omega_0^2\right]Y(s)=\omega_0 X(s)$$

$$\left[s^2+\frac{2s}{\tau}+\omega_0^2+\frac{1}{\tau^2}\right]Y(s)=\omega_0 X(s)$$

$$\left[\frac{d^2}{dt^2}+\frac{2}{\tau}\frac{d}{dt}+\omega_0^2+\frac{1}{\tau^2}\right]y(t)=\omega_0 x(t)$$

Physical systems?

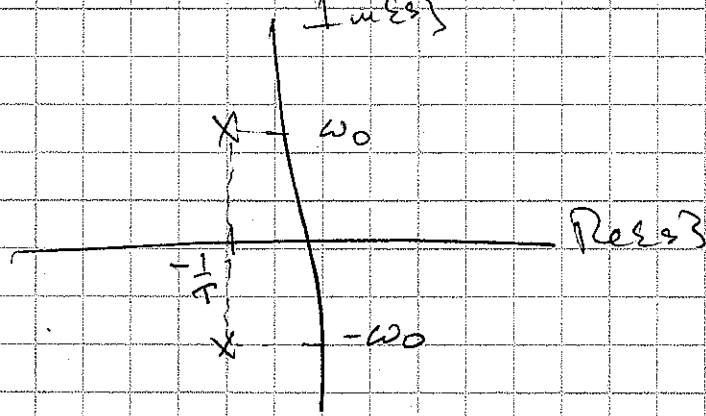
Mass + Spring with damping

RLC circuit

Lots and lots of others

(Make sure to tell the students that we started with the physical examples...)

Poles and zeroes



Fourier transform

$$H(j\omega) = \frac{\omega_0}{\left(j\omega + \frac{1}{\tau}\right)^2 + \omega_0^2}$$

$$= \frac{\omega_0}{\omega_0^2 + \frac{1}{\tau^2} - \omega^2 + 2\frac{j\omega}{\tau}}$$

$$= -\frac{\omega_0}{\omega^2 - 2\frac{j\omega}{\tau} - \omega_0^2 - \frac{1}{\tau^2}}$$