6.003R

March 12, 2010

Example

Pat Hagelt

$$X \longrightarrow \boxed{H} \longrightarrow Y$$

let $h(t) = \sin(\omega_0 t)\, e^{-t/\tau} u(t)$

What is $H(s)$?

$$H(s) = \int_{-\infty}^{\infty} h(t) e^{-st} dt$$

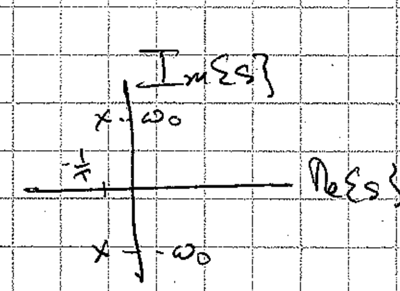
$$= \int_{-\infty}^{\infty} \sin(\omega_0 t)\, e^{-t/\tau} u(t) e^{-st} dt$$

$$= \int_{0}^{\infty} e^{-(s+\frac{1}{\tau})t} \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j} dt$$

$$= \frac{1}{2j} \int_{0}^{\infty} e^{-(s+\frac{1}{\tau}-j\omega_0)t} - e^{-(s+\frac{1}{\tau}+j\omega_0)t} dt$$

$$= \frac{1}{2j}\left[\frac{1}{s+\frac{1}{\tau}-j\omega_0} - \frac{1}{s+\frac{1}{\tau}+j\omega_0} \right] \qquad Re\{s\} > -\frac{1}{\tau}$$

$$= \frac{\omega_0}{(s+\frac{1}{\tau})^2 + \omega_0^2}$$



What is the associated differential equation?

$$Y(s) = H(s)\, X(s)$$

$$= \frac{\omega_0}{\left(s+\frac{1}{\tau}\right)^2 + \omega_0^2}\, X(s)$$

$$\left[\left(s+\frac{1}{\tau}\right)^2 + \omega_0^2\right] Y(s) = \omega_0 X(s)$$

$$\downarrow$$

$$\left[\left(\frac{d}{dt} + \frac{1}{\tau}\right)^2 + \omega_0^2\right] y(t) = \omega_0 x(t)$$

$$\frac{d^2}{dt^2} y(t) + \frac{2}{\tau}\frac{d}{dt} y(t) + \frac{1}{\tau^2} y(t) + \omega_0^2 y(t) = \omega_0 x(t)$$

Models?

Mass + spring with damping

RLC circuit

Lossy E+M cavity

Many others

Fourier transform

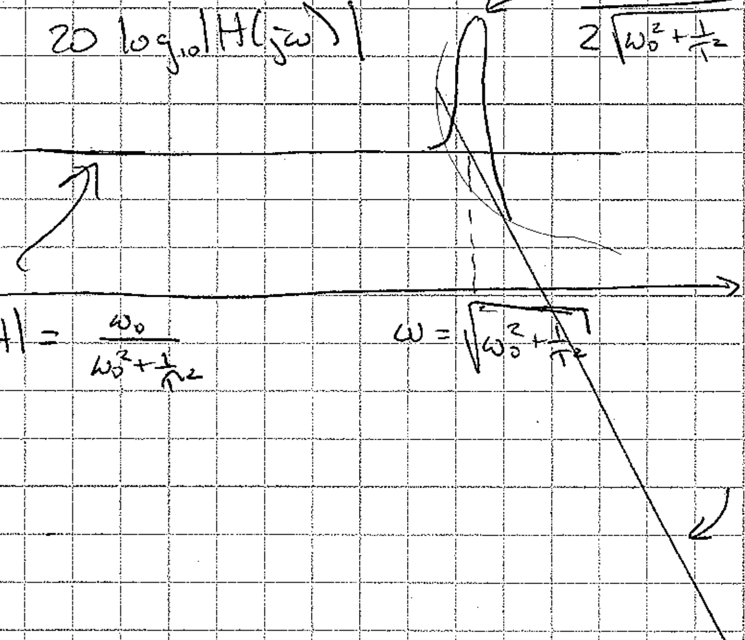
$$H(j\omega) = \frac{\omega_0}{\left(j\omega + \frac{1}{\tau}\right)^2 + \omega_0^2}$$

$$= \frac{\omega_0}{-\omega^2 + 2\frac{j\omega}{\tau} + \frac{1}{\tau^2} + \omega_0^2}$$

Bode plot

$20 \log_{10}|H(j\omega)|$

$\frac{\omega_0 \tau}{2\sqrt{\omega_0^2 + \frac{1}{\tau^2}}} = |H|$



$\omega, \log$

$|H| = \frac{\omega_0}{\omega_0^2 + \frac{1}{\tau^2}}$

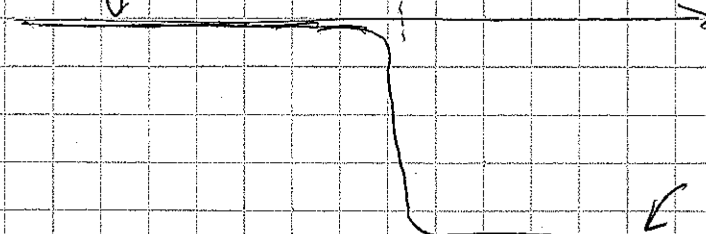
$\omega = \sqrt{\omega_0^2 + \frac{1}{\tau^2}}$

$|H| \rightarrow \frac{\omega_0}{\omega^2}$

$\angle H(j\omega)$

$\angle H(j\omega) \rightarrow 0$

$\omega = \sqrt{\omega_0^2 + \frac{1}{\tau^2}}$



$\omega, \log$

$\angle H(j\omega) \rightarrow -\pi$

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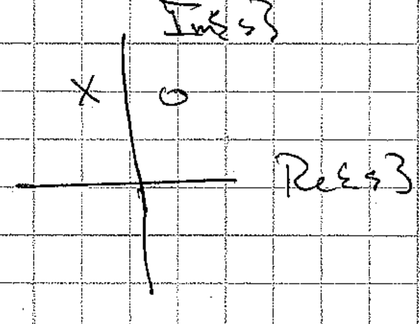
Example

Want an impulse response that corresponds to



From before, we recall that

$$h(t) = e^{j\omega_0 t} e^{-t/\tau} u(t)$$

Resulted in

$$H(s) = \frac{1}{s + \frac{1}{\tau} - j\omega_0}$$

So we now want

$$H_1(s) = \frac{s - \frac{1}{\tau} - j\omega_0}{s + \frac{1}{\tau} - j\omega_0}$$

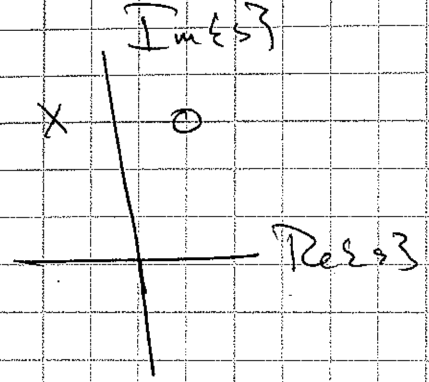
$$= \left(s - \frac{1}{\tau} - j\omega_0\right)\left(s + \frac{1}{\tau} - j\omega_0\right)$$

$$h_1(t) = \left(\frac{d}{dt} - \frac{1}{\tau} - j\omega_0\right) h(t)$$

$$= \left(j\omega_0 - \frac{1}{\tau}\right) h(t) + \delta(t) - \left(j\omega_0 + \frac{1}{\tau}\right) h(t)$$

$$= \delta(t) - \frac{2}{\tau} h(t)$$

B1

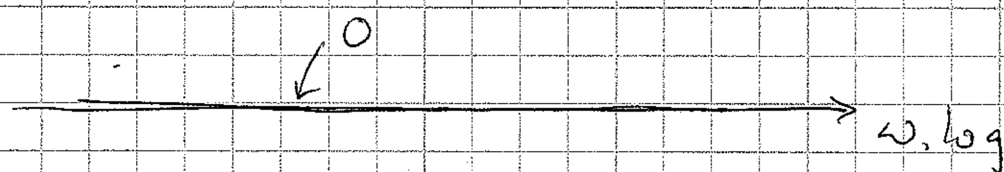


$$H_1(s) = \frac{s - \frac{1}{\tau} - j\omega_0}{s + \frac{1}{\tau} - j\omega_0} \qquad h_1(t) = \delta(t) - \frac{2}{\tau} e^{j\omega_0 t} e^{-t/\tau} u(t)$$

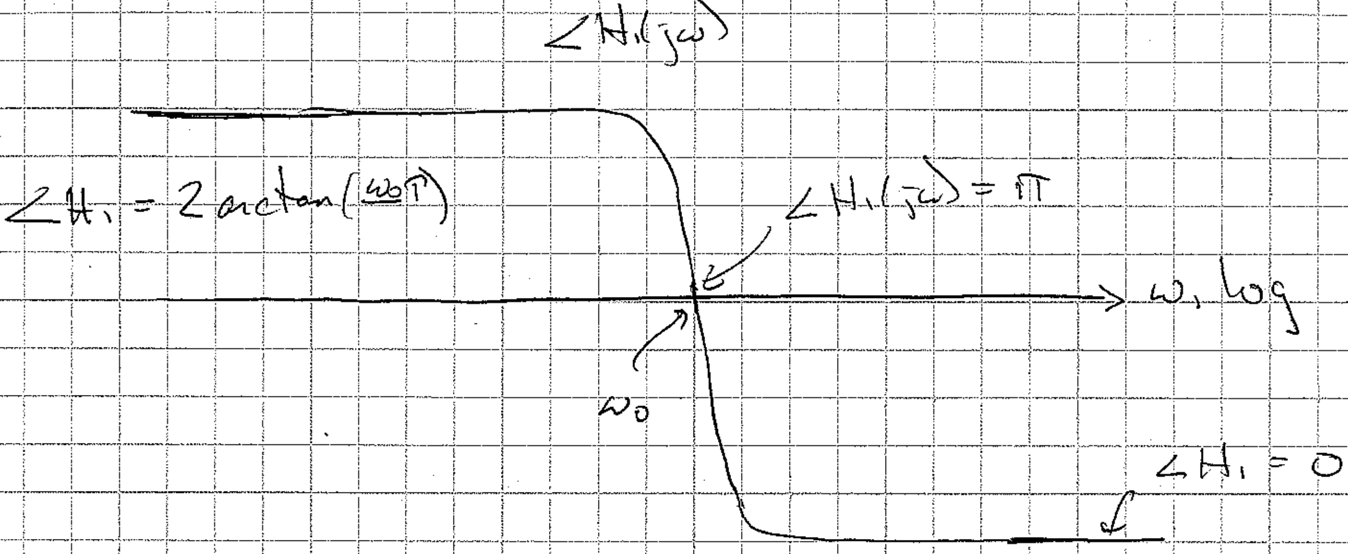
Any insight or intuition about $h_1(t)$ relative to pole-zero plot? Not much. The zero means take a derivative and add to a complex multiple of old $h(t)$.

<u>Bode plot</u>

$$20 \log_{10} |H_1(j\omega)|$$



$$H_1(j\omega) = \frac{j(\omega - \omega_0) - \frac{1}{\tau}}{j(\omega - \omega_0) + \frac{1}{\tau}} \qquad |H_1| = \frac{\sqrt{(\omega - \omega_0)^2 + \frac{1}{\tau^2}}}{\sqrt{(\omega - \omega_0)^2 + \frac{1}{\tau^2}}} = 1$$



Angle is just 2x the result from the pole alone

B3

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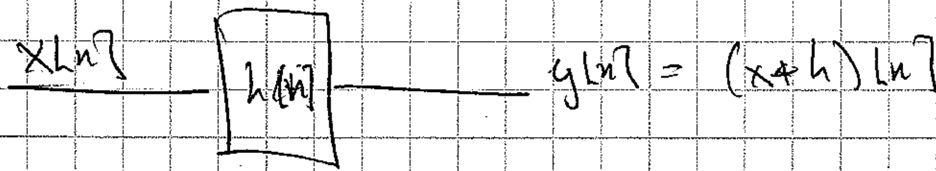
Example: Discrete system



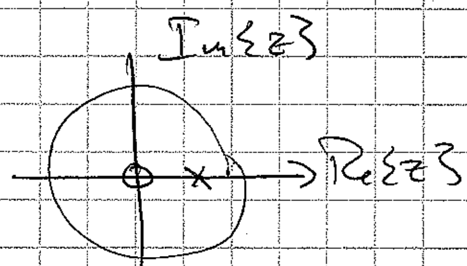
Try simplest nontrivial h[n]:

$$h[n] = \alpha^n u[n]$$

$$H(z) = \sum_{n=-\infty}^{\infty} \alpha^n u[n] z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{\alpha}{z}\right)^n = \frac{1}{1 - \frac{\alpha}{z}} \qquad \left|\frac{\alpha}{z}\right| > 1$$

$$= \frac{z}{z-\alpha}$$



Difference eqn

$$Y(z) = H(z) X(z) = \frac{z}{z-\alpha} X(z)$$

$$(z-\alpha) Y(z) = z X(z)$$

C1

$$y[n+1] - \alpha y[n] = x[n+1]$$

or

$$y[n] = \alpha y[n-1] + x[n]$$

Models:

Bank account

others

Example: Oscillatory discrete system

$$\xrightarrow{x[n]} \boxed{h[n]} \longrightarrow y[n] = (x * h)[n]$$

$$h[n] = \alpha^n e^{j\Omega_0 n} u[n]$$

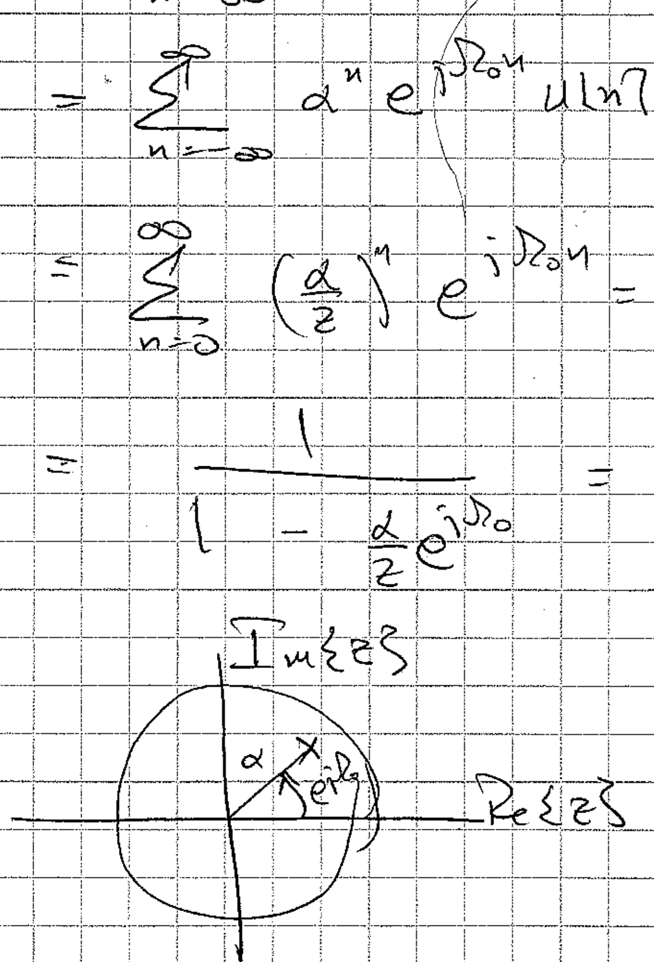
What is $H(z)$?

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \alpha^n e^{j\Omega_0 n} u[n] z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{\alpha}{z}\right)^n e^{j\Omega_0 n} = \sum_{n=0}^{\infty} \left(\frac{\alpha e^{j\Omega_0}}{z}\right)^n$$

$$= \frac{1}{1 - \frac{\alpha}{z} e^{j\Omega_0}} = \frac{z}{z - \alpha e^{j\Omega_0}} \qquad \left|\frac{\alpha}{z}\right| < 1$$



D1

Difference equation

$$Y(z) = \frac{z}{z - \alpha e^{j\Omega_0}} X(z)$$

$$\left(z - \alpha e^{j\Omega_0}\right) Y(z) = z X(z)$$

$$y[n+1] - \alpha e^{j\Omega_0} y[n] = x[n+1]$$

or

$$y[n] = \alpha e^{j\Omega_0} y[n-1] + x[n]$$

No obvious associated simple model, but interesting nonetheless ...

Example

$$\xrightarrow{x[n]} \boxed{h[n]} \longrightarrow y[n] = (x*h)[n]$$

$$h[n] = \sin(\Omega_0 n)\, \alpha^n\, u[n]$$

What is $H(z)$?

$$H(z) = \sum_{n=-\infty}^{\infty} \sin(\Omega_0 n)\, \alpha^n\, u[n]\, z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{\alpha}{z}\right)^n \frac{e^{j\Omega_0 n} - e^{-j\Omega_0 n}}{2j}$$

$$= \frac{1}{2j} \sum_{n=0}^{\infty} \left(\frac{\alpha e^{j\Omega_0}}{z}\right)^n - \left(\frac{\alpha e^{-j\Omega_0}}{z}\right)^n$$

$$= \frac{1}{2j} \left\{ \frac{1}{1 - \frac{\alpha}{z} e^{j\Omega_0}} - \frac{1}{1 - \frac{\alpha e^{-j\Omega_0}}{z}} \right\}$$

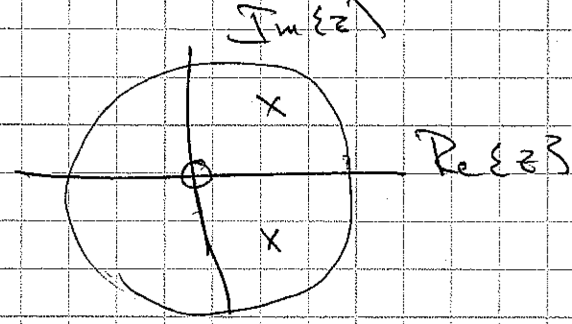
$$= \frac{1}{2j} \left\{ \frac{z}{z - \alpha e^{j\Omega_0}} - \frac{z}{z - \alpha e^{j\Omega_0}} \right\}$$

$$= \frac{1}{2j} \left\{ \frac{z^2 - z\alpha e^{-j\Omega_0} - z^2 + z\alpha e^{j\Omega_0}}{z^2 - \alpha z\left(e^{j\Omega_0} + e^{-j\Omega_0}\right) + \alpha^2} \right\}$$

F1

$$= \frac{1}{2j} \frac{z\alpha\left(e^{j\Omega_0} - e^{-j\Omega_0}\right)}{z^2 - 2\alpha z\cos\Omega_0 + \alpha^2}$$

$$= \frac{z\alpha\sin\Omega_0}{z^2 - 2\alpha z\cos\Omega_0 + \alpha^2}$$



What is diff eqn?

$$H(z) = \frac{z\alpha\sin\Omega_0}{z^2 - 2\alpha z\cos\Omega_0 + \alpha^2}$$

$$Y(z) = H(z)X(z)$$

$$\left(z^2 - 2\alpha z\cos\Omega_0 + \alpha^2\right)Y(z) = z\alpha\sin\Omega_0 X(z)$$

$$y[n+2] - 2\alpha\cos\Omega_0 y[n+1] + \alpha^2 y[n] = \alpha\sin\Omega_0 x[n+1]$$

shift

$$y[n] = 2\alpha\cos\Omega_0 y[n-1] - \alpha^2 y[n-2] + \alpha\sin\Omega_0 x[n]$$

F2

Example

Convolution of 2 oscillatory discrete functions

$$x[n] = \cos(\Omega_0 n)\, \alpha^n\, u[n]$$

$$y[n] = \cos(\Omega_1 n)\, \beta^n\, u[n]$$

$$(x * y)[n] = \sum_{k=-\infty}^{\infty} \cos(\Omega_0 k)\, \alpha^k\, u[k] \times \cos(\Omega_1 (n-k))\, \beta^{n-k}\, u[n-k]$$

$$= \sum_{k=0}^{\infty} \cos(\Omega_0 k)\, \alpha^k \cos(\Omega_1 (n-k))\, \beta^{n-k}\, u[n-k]$$

$$= u[n] \sum_{k=0}^{n} \cos(\Omega_0 k) \cos \Omega_1 (n-k) \left(\frac{\alpha}{\beta}\right)^k \beta^n$$

$$= \beta^n u[n] \sum_{k=0}^{n} \left(\frac{\alpha}{\beta}\right)^k \left(\frac{e^{j\Omega_0 k} + e^{-j\Omega_0 k}}{2}\right) \left(\frac{e^{j\Omega_1 (n-k)} + e^{-j\Omega_1 (n-k)}}{2}\right)$$

$$= \frac{\beta^n u[n]}{4} \sum_{k=0}^{n} \left(\frac{\alpha}{\beta}\right)^k \Big(e^{j(\Omega_0 - \Omega_1)k} e^{j\Omega_1 n} + e^{j(\Omega_0 + \Omega_1)k} e^{-j\Omega_1 n}$$

$$+ e^{-j(\Omega_0 + \Omega_1)k} e^{j\Omega_1 n} + e^{-j(\Omega_0 - \Omega_1)k} e^{-j\Omega_1 n} \Big)$$

E1

$$= \frac{1}{4}\,\beta^n u[n] \sum_{k=0}^{n} \left[\frac{\alpha}{\beta} e^{j(\Omega_0-\Omega_1)}\right]^k e^{j\Omega_1 n}$$

$$+ \frac{1}{4}\,\beta^n u[n] \sum_{k=0}^{n} \left[\frac{\alpha}{\beta} e^{j(\Omega_0+\Omega_1)}\right]^k e^{-j\Omega_1 n}$$

$$+ \frac{1}{4}\,\beta^n u[n] \sum_{k=0}^{n} \left[\frac{\alpha}{\beta} e^{-j(\Omega_0+\Omega_1)}\right]^k e^{j\Omega_1 n}$$

$$+ \frac{1}{4}\,\beta^n u[n] \sum_{k=0}^{n} \left[\frac{\alpha}{\beta} e^{-j(\Omega_0-\Omega_1)}\right]^k e^{-j\Omega_1 n}$$

$$= \frac{1}{4}\,\beta^n u[n] \left[\frac{1 - \left(\frac{\alpha}{\beta}\right)^{n+1} e^{j(\Omega_0-\Omega_1)(n+1)}}{1 - \frac{\alpha}{\beta} e^{j(\Omega_0-\Omega_1)}}\right] e^{j\Omega_1 n}$$

$$+ \frac{1}{4}\,\beta^n u[n] \left[\frac{1 - \left(\frac{\alpha}{\beta}\right)^{n+1} e^{j(\Omega_0+\Omega_1)(n+1)}}{1 - \left(\frac{\alpha}{\beta}\right) e^{j(\Omega_0+\Omega_1)}}\right] e^{-j\Omega_1 n}$$

$$+ \frac{1}{4}\,\beta^n u[n] \left[\frac{1 - \left(\frac{\alpha}{\beta}\right)^{n+1} e^{-j(\Omega_0+\Omega_1)(n+1)}}{1 - \frac{\alpha}{\beta} e^{-j(\Omega_0+\Omega_1)}}\right] e^{j\Omega_1 n}$$

$$+ \frac{1}{4}\,\beta^n u[n] \left[\frac{1 - \left(\frac{\alpha}{\beta}\right)^{n+1} e^{-j(\Omega_0-\Omega_1)(n+1)}}{1 - \frac{\alpha}{\beta} e^{-j(\Omega_0-\Omega_1)}}\right] e^{-j\Omega_1 n}$$

$$= \frac{1}{4} u[n] \left(\frac{\beta^{n+1} e^{j\Omega_1 (n+1)} - \alpha^{n+1} e^{j\Omega_0 (n+1)}}{\beta e^{j\Omega_1} - \alpha e^{j\Omega_0}} \right)$$

$$+ \frac{1}{4} u[n] \left(\frac{\beta^{n+1} e^{-j\Omega_1 (n+1)} - \alpha^{n+1} e^{j\Omega_0 (n+1)}}{\beta e^{-j\Omega_1} - \alpha e^{j\Omega_0}} \right)$$

$$+ \frac{1}{4} u[n] \left(\frac{\beta^{n+1} e^{j\Omega_1 (n+1)} - \alpha^{n+1} e^{-j\Omega_0 (n+1)}}{\beta e^{j\Omega_1} - \alpha e^{-j\Omega_0}} \right)$$

$$+ \frac{1}{4} u[n] \left(\frac{\beta^{n+1} e^{-j\Omega_1 (n+1)} - \alpha^{n+1} e^{-j\Omega_0 (n+1)}}{\beta e^{-j\Omega_1} - \alpha e^{-j\Omega_0}} \right)$$

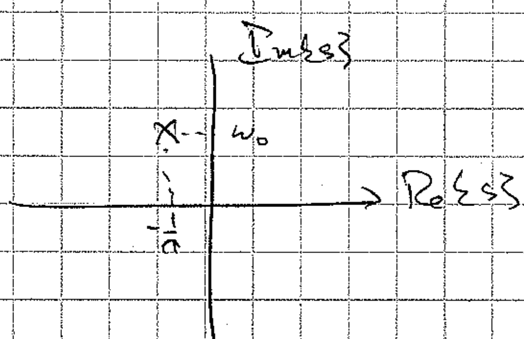
when is this maximized?

63

Driving a system on or near Resonance

$$X \longrightarrow \boxed{H} \longrightarrow Y \qquad\qquad Y = H X$$

Suppose that $H(s)$ has a pole at $-\frac{1}{\tau} + j\omega_0$



$$H(s) = \frac{1}{s + \frac{1}{\tau} - j\omega_0} \qquad\qquad h(t) = e^{-t/\tau} e^{j\omega_0 t} u(t)$$

What is the response to steady-state (eternal) signal

$$x(t) = e^{j\omega t} \quad ?$$

Laplace transform $X(s)$ has no region of convergence.

Instead use eigenvalue relation

$$y(t) = H(s)\Big|_{s=j\omega} x(t) = \frac{1}{j(\omega-\omega_0) + \frac{1}{\tau}} \underbrace{e^{j\omega t}}_{x(t)}$$

#1

So, what happens if

$$x(t) = u(t) e^{j\omega t}$$

Now, this has Laplace transform

$$\overline{X}(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$= \int_{-\infty}^{\infty} e^{j\omega t} u(t) e^{-st} dt$$

$$= \int_{0}^{\infty} e^{-(s-j\omega)t} dt$$

$$= \frac{1}{s - j\omega} \qquad \mathrm{Re}\{s\} > 0$$

So $\quad Y(s) = H(s)\overline{X}(s)$

$$= \frac{1}{s+\frac{1}{\tau}-j\omega_0} \cdot \frac{1}{s-j\omega}$$

Can we recover $y(t)$? At least 2 ways —

partial fractions

convolution

H2

Look at convolution

$$y(t) = (x * h)(t)$$

$$= \int_{-\infty}^{\infty} x(t')\, h(t-t')\, dt'$$

$$= \int_{-\infty}^{\infty} e^{j\omega t} u(t)\; e^{-(t-t')/\tau}\, e^{j\omega_0(t-t')}\, u(t-t')\, dt'$$

$$= \int_{0}^{\infty} e^{j\omega t}\, e^{-(t-t')/\tau}\, e^{j\omega_0(t-t')}\, u(t-t')\, dt'$$

$$= u(t) \int_{0}^{t} e^{j(\omega-\omega_0)t'}\, e^{t'/\tau}\, dt'\; e^{-t/\tau}\, e^{j\omega_0 t}$$

$$= u(t)\, e^{j\omega_0 t}\, e^{-t/\tau} \left. \frac{e^{t'/\tau}\, e^{j(\omega-\omega_0)t'}}{\frac{1}{\tau} + j(\omega-\omega_0)} \right|_{0}^{t}$$

$$= u(t)\, e^{j\omega_0 t}\, e^{-t/\tau} \left[\frac{e^{t/\tau}\, e^{j(\omega-\omega_0)t} - 1}{\frac{1}{\tau} - j(\omega-\omega_0)} \right]$$

$$= u(t) \left\{ \frac{e^{j\omega t} - e^{j\omega_0 t}\, e^{-t/\tau}}{\frac{1}{\tau} - j(\omega-\omega_0)} \right\}$$

terms cancel approx for $\frac{t}{\tau} < 1$

Biggest signal when $\omega = \omega_0$

H3