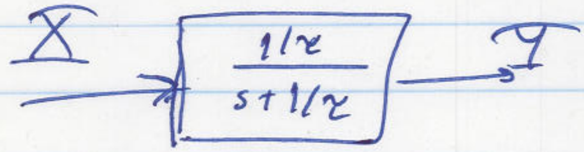
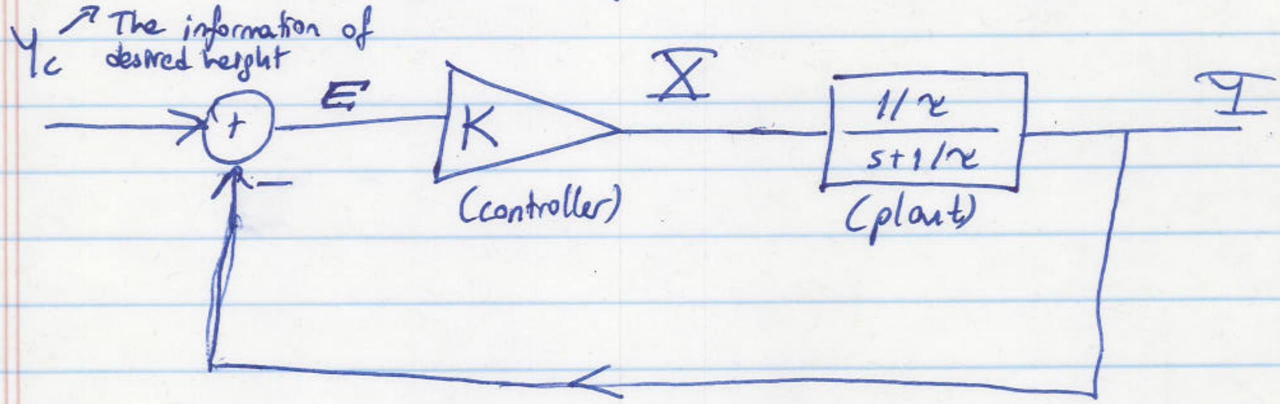
From Wednesday:

Controlling a 1^{st} order system with different feedback schemes



- First method: Proportional control
  - Controller stage is just a "gain" stage.



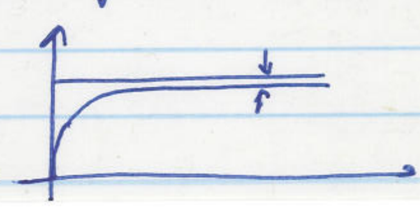
- E is error in position/height. Error is multiplied by K. K can be regarded as "how much we care about error".
- Speed of filling rate is determined by "how far are we from the desired height?". It makes sense to fill at a higher rate if we are far from desired height Y_c.
- There is always going to be steady state error if we want a height other than 0. (because $E = \frac{Y}{K \cdot H(s)} \neq 0$)
- We found

$$H(s) = \frac{Y(s)}{Y_c(s)} = \frac{K/\tau}{s + \frac{1}{\tau} + \frac{K}{\tau}}.$$

The steady state value of Y if $Y_c = u(t)$ is

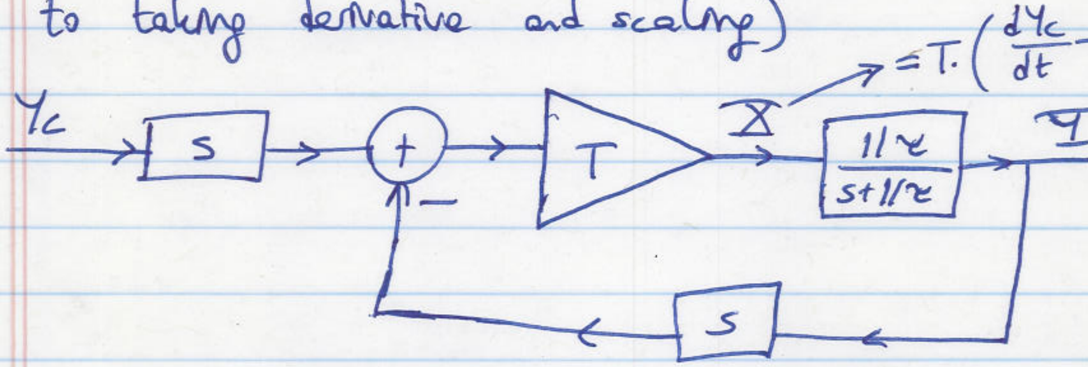
$$\lim_{s \to 0} s \cdot \frac{1}{s} \cdot \frac{K/\tau}{s + 1/\tau + K/\tau} = \frac{K}{K+1} < 1.$$

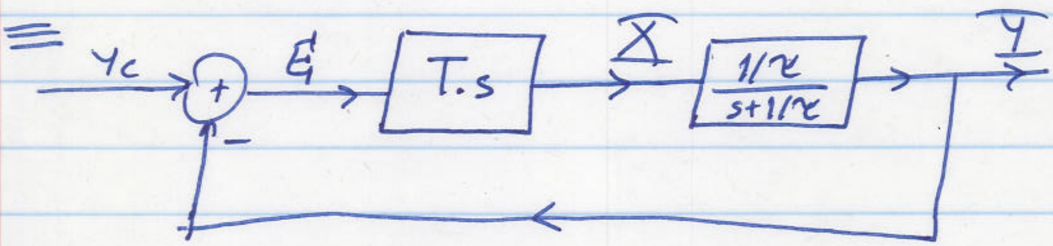
makes sense to be lower than 1.



- Second method: Derivative control

– Controller stage is not "gain", instead $T.s$ (which corresponds to taking derivative and scaling)





Analogy of blind man looking for his dog in a room (can find only if dog moves) $\Leftarrow$

– E is the error in position/height. We are driving the plant with the information of difference between desired rate of change in height and actual rate of change in height. T can be regarded as "how much we care about difference in rate of changes".

– If we want the height to change slowly ($\frac{dY_c}{dt}$ is low) but Y itself is changing fast ($\frac{dY}{dt}$ is large), the feedback will tell the plant to "slow down" by decreasing X.

– Looks fancy but actually bears a potential danger: What if Y_c is a constant and Y is settled at a different level than Y_c? No change in Y will be observed since X is 0. But this also means $Y=0$ because $Y = H(s).X = 0$. For every slow varying desired level Y_c, we have a potential risk of settling at 0 at the output!

$$H(s) = \frac{Y(s)}{Y_c(s)} = \frac{(T/\tau)s}{s + \frac{1}{\tau} + (T/\tau)s}$$. Unit step response at $t=\infty$ is 0!

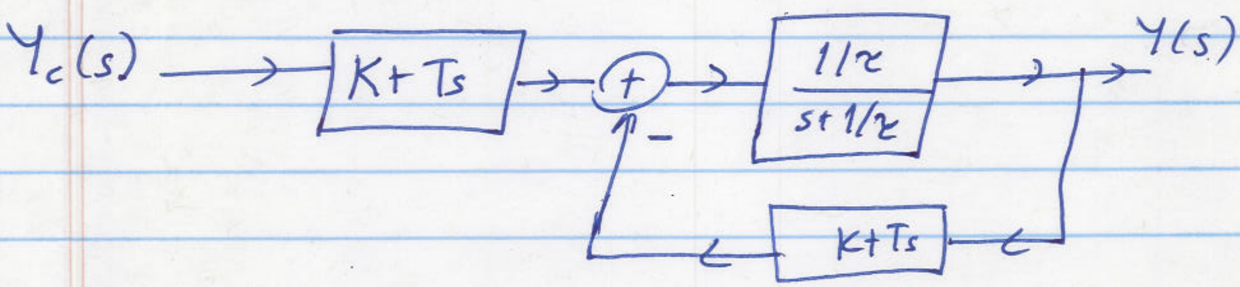
Why not have the information of "actual level" and "desired rate of height change"

- Third method: Proportional and Derivative Control
- Controller includes both gain and derivative

Let

$$x(t) = K[y_c(t) - y(t)] + T\left(\frac{dy_c(t)}{dt} - \frac{dy(t)}{dt}\right)$$

Corresponding feedback system:



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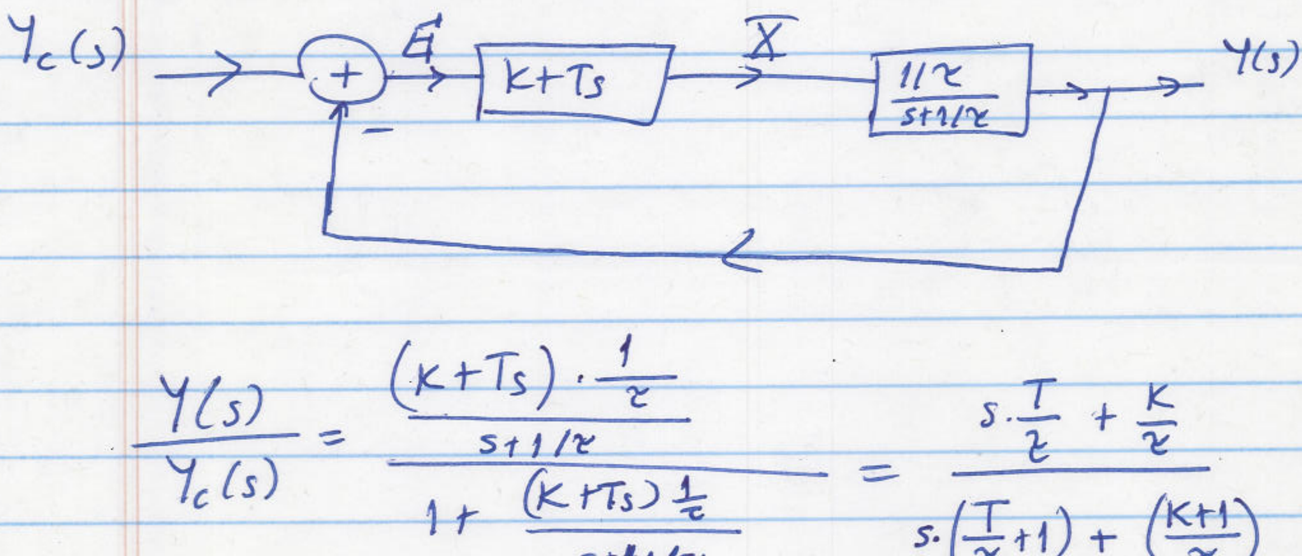


$$\frac{Y(s)}{Y_c(s)} = \frac{\dfrac{(K+Ts)\cdot\frac{1}{\tau}}{s+1/\tau}}{1 + \dfrac{(K+Ts)\frac{1}{\tau}}{s+1/\tau}} = \frac{s\cdot\frac{T}{\tau} + \frac{K}{\tau}}{s\cdot\left(\frac{T}{\tau}+1\right) + \left(\frac{K+1}{\tau}\right)}$$

Let's quickly check the unit step response from initial and final value theorems:

$t_\infty \Rightarrow \lim_{s\to 0} s\cdot\frac{1}{s}\cdot H(s) = \frac{K}{K+1}$ We need large K

$t_0 \Rightarrow \lim_{s\to\infty} s\cdot\frac{1}{s}\cdot H(s) = \frac{T}{T+\tau}$ We need large T as well

Try $K = 100$ and $\frac{T}{\tau} = 100$

$$H(s) = \frac{\frac{100}{\tau} + 100s}{101s + \frac{101}{\tau}} = \frac{100\cancel{(s+\frac{1}{\tau})}}{101\cancel{(s+\frac{1}{\tau})}} = \frac{100}{101}$$

Not dependent on s anymore! H(s) is just the same factor for all s. We can correct the gain by another gain stage, which is trivial.

- Feedback here seems to be able to "undo" what the differential equation does.

One issue: Pole zero approximate cancellation:
In a real system, pole-zero cancellation might not be exact. Suppose that

$$K = 100 \quad \text{and} \quad \frac{T}{\tau} = 99$$

This results in $H(s) = \dfrac{\frac{100}{\tau} + 99s}{100s + 101/\tau} = \dfrac{s + \frac{100}{99}\cdot\frac{1}{\tau}}{s + \frac{101}{100}\cdot\frac{1}{\tau}}$

No cancellation! The pole is located at $-\frac{101}{100\tau}$ and the zero is located at

$-\frac{100}{99\tau}$.

Look at step response for imperfect pole-zero cancellation

$$H(s) = \frac{s\cdot\frac{T}{\tau} + \frac{K}{\tau}}{s\cdot\left(\frac{T}{\tau}+1\right) + \frac{K+1}{\tau}} = \frac{\frac{T}{\tau}}{\frac{T}{\tau}+1} \cdot \frac{s + \frac{K}{T}}{s + \frac{K+1}{T+\tau}}$$

$$= \frac{T}{T+\tau} \cdot \left(\frac{s + \frac{K+1}{T+\tau} + \underbrace{\left(\frac{K}{T} - \frac{K+1}{T+\tau}\right)}_{\text{call this } \alpha}}{s + \underbrace{\frac{K+1}{T+\tau}}_{\text{call this } \beta}} \right)$$

$$= \frac{T}{T+\tau}\left(1 + \frac{\alpha}{s+\beta}\right)$$ (Instead of a constant, additional term)

If $y_c(t) = u(t) \Rightarrow Y_c(s) = \frac{1}{s} \Rightarrow Y(s) = \frac{1}{s}\cdot\left(\frac{T}{T+\tau} + \frac{\alpha}{s+\beta}\right)$

~~y(t)~~

$$Y(s) = \frac{1}{s}\cdot\frac{T}{T+\tau} + \frac{\alpha}{s(s+\beta)}$$

$$\frac{A}{\underset{(s+\beta)}{s}} + \frac{B}{\underset{(s)}{s+\beta}} = \frac{\alpha}{s(s+\beta)} \Rightarrow (A+B)s + \beta A = \alpha$$

$A = -B = +\frac{\alpha}{\beta}$

$A = \frac{\alpha}{\beta}$

$B = -\frac{\alpha}{\beta}$

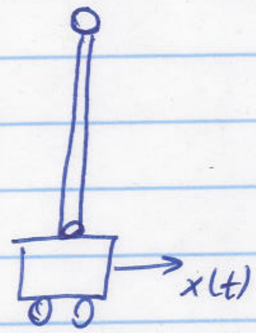
$$Y(s) = \frac{1}{s}\left(\frac{T}{T+\tau} + \frac{\alpha}{\beta}\right) - \frac{\alpha/\beta}{s+\beta}$$

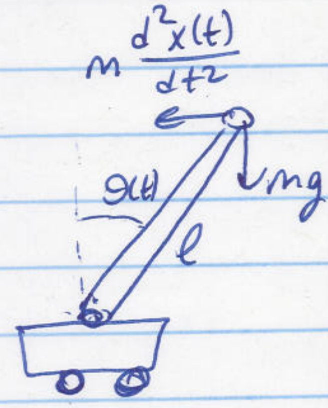
$$\frac{\alpha}{\beta} = \frac{K}{K+1}\cdot\frac{T+\tau}{T} - 1 \approx 0$$

Moreover, $y(t) = \left(\underbrace{\frac{T}{T+\tau}}_{\approx 1} + \underbrace{\frac{\alpha}{\beta}}_{\text{very small}}\right) u(t) - \underbrace{\frac{\alpha}{\beta}}_{\text{very small}}\cdot e^{-\beta t} u(t)$

decay rate depends on placement of the pole (T & K)

Inverted Pendulum:





$$\underbrace{ml^2}_{\text{moment of inertia}} \underbrace{\frac{d^2\theta(t)}{dt^2}}_{\text{angular acceleration}} = \underbrace{mg\,l\sin\theta(t)}_{\text{torque due to gravitation}} - \underbrace{m\frac{d^2x(t)}{dt^2}\,l\cos\theta(t)}_{\text{Torque due to acceleration of car}}$$

$$\vec{\tau} = I\cdot\vec{\alpha} \quad \text{(Newton's second law for rotation)}$$

Small angle approximation: (We are interested only around $\theta = 0$)

$$ml^2\frac{d^2\theta(t)}{dt^2} = mgl\,\theta(t) - m\frac{d^2x(t)}{dt^2}l \qquad \text{since } \sin\theta \approx \theta \text{ and } \cos\theta \approx 1 \text{ for small } \theta$$

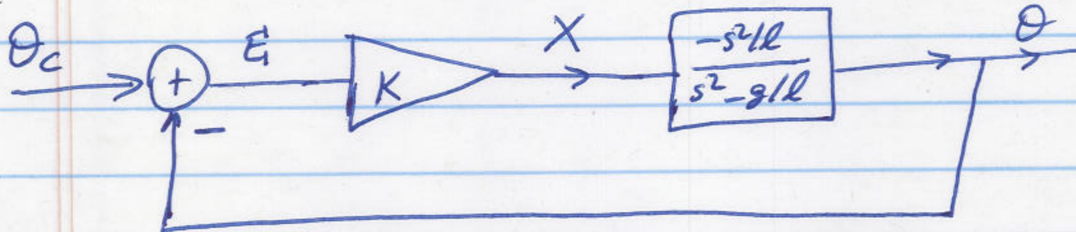
$$ml^2s^2\theta(s) = mgl\,\theta(s) - ms^2 l\,X(s)$$

$$\frac{\theta(s)}{X(s)} = \frac{-ms^2l}{ml^2s^2 - mgl} = \frac{-s^2/l}{s^2 - g/l}$$

Clearly, the system is unstable as it is since one of the poles is on the Right Half Plane (RHP). $s = \pm\sqrt{g/l}$

Try using proportional control to stabilize:

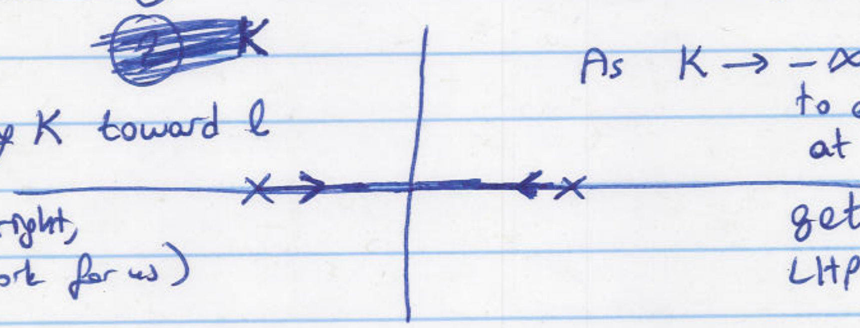
- We want to use a "proportion" (might be larger than 1) of the error between actual θ and the desired θ_c to tell our motor how much and in which direction to move the car so that θ stays around θ_c (= 0 in our case)



$$\frac{\theta}{\theta_c} = \frac{-\dfrac{K s^2/l}{s^2-g/l}}{1-\dfrac{K s^2/l}{s^2-g/l}} = \frac{-K s^2/l}{\left(1-\frac{K}{l}\right)s^2-\frac{g}{l}}$$

Roots: $s = \pm\sqrt{\dfrac{g/l}{1-K/l}}$

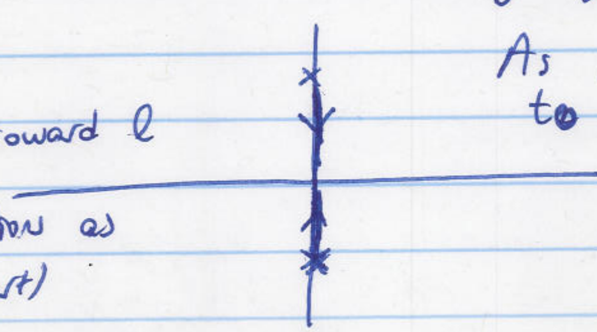
Two possibilities: ① $K < l$: Two real roots, one on RHP, one on LHP



(Increasing K toward l will push the zero even more to the right, which won't work for us)

As $K \to -\infty$, roots get closer to each other and meet at 0. We cannot get the RHP to the LHP with this option.

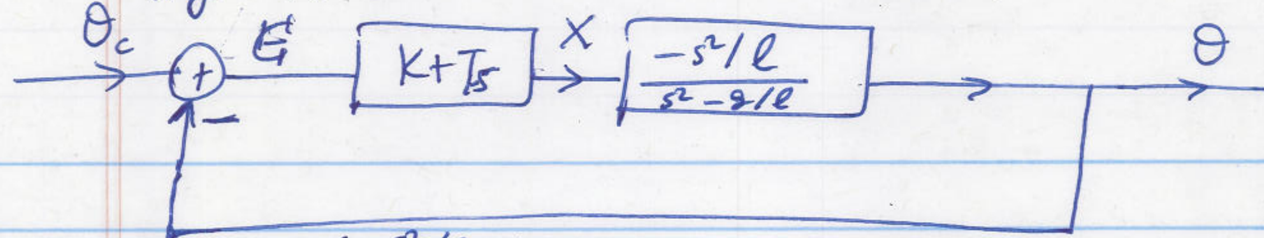
② $K > l$: Two imaginary roots on $j\omega$ axis.



Decreasing K toward l will increase the frequency of oscillation as it moves poles apart)

As $K \to \infty$, roots get closer to each other and meet at 0. We always have oscillations for this case.

Try PD control (momentum involved)



$$\frac{\theta}{\theta_c} = \frac{(K+T \cdot s)\left(\frac{-s^2/l}{s^2-g/l}\right)}{1+\frac{-(K+T\cdot s)s^2/l}{s^2-g/l}} = \frac{(K+T\cdot s)\cdot(-s^2/l)}{s^2-\frac{g}{l}-(K+Ts)s^2/l}$$

Pay attention to the denominator only:

$$-\frac{Ts^3}{l} - \left(\frac{K}{l}-1\right)s^2 - \frac{g}{l} = 0 \qquad \frac{g}{l} \approx \frac{10}{0{,}5} = 20$$

$$\frac{T}{l}s^3 + \left(\frac{K}{l}-1\right)s^2 + 20 = 0$$

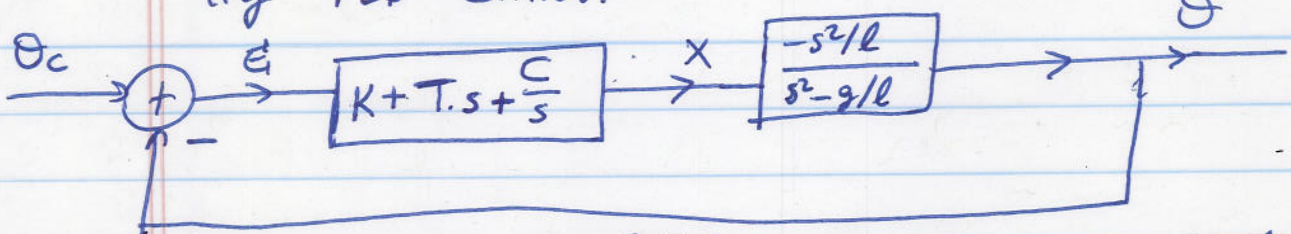
Routh – Hurwitz method: A stability criterion often used in control theory. Checks on the relationship between coefficients of a polynomial and tells you how many roots are on the RHP. For a polynomial of the above form, there is always a root on RHP, so this system is still unstable.

All these terms should have the same sign for stability

$\frac{T}{l}$	0	0
$\left(\frac{K}{l}-1\right)$	20	0
$\frac{-20T/l}{\frac{K}{l}-1}$	0	
20		

$\Longrightarrow$ $\frac{T}{l}$ and $\frac{K}{l}-1$ have to have the same signs, which means the 3rd term will be opposite sign and there is a RHP pole!

Try PID control:



$$\frac{\theta}{\theta_c} = \frac{\left(K + T.s + \frac{C}{s}\right)\cdot\left(\frac{-s^2/l}{s^2 - g/l}\right)}{1 + \frac{-s^2/l\,\left(K + T.s + \frac{C}{s}\right)}{s^2 - g/l}} = \frac{sK + Ts^2 + \frac{C}{s}\left(\frac{-s^2/l}{s^2 - g/l}\right)}{s^2 - \frac{g}{l} - s^2\cdot\frac{K}{l} - \frac{T}{l}s^3 - \frac{C}{l}s}$$

Pay attention to the denominator (making sure numerator doesn't have extra s in the bottom)

$$\frac{T}{l}s^3 + \left(\frac{K}{l} - 1\right)s^2 + \frac{C}{l}s + \underbrace{\frac{g}{l}}_{\approx 20} = 0$$

A similar analysis with Routh–Hurwitz yields all LHP poles for $\frac{T}{l} > 0$, $\frac{K}{l} - 1 > 0$ and $\left(\frac{K}{l} - 1\right)\cdot\frac{C}{l} > 20\,\frac{T}{l}$.

For example, (assuming $l = 0,5$) $T = 5$, $K = 5,5$, $C = 20$ yields 3 poles:

$$\left.\begin{aligned} s &= -0,53 \\ s &= j1,92 - 0,23 \\ s &= -j1,92 - 0,23 \end{aligned}\right\}\ \text{all on LHP.}$$

We have finally stabilized the system!