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Expansion in Hilbert space (Advanced)

Basic idea: expand a function in a set of ortho-normal basis functions

$$f(x) = \sum_j a_j \phi_j(x)$$

Ortho-normal basis functions satisfy

$$\int_a^b \phi_k^*(x)\, \phi_j(x)\, dx = \begin{cases} 0 & j \neq k \\ 1 & j = k \end{cases}$$

This allows one to determine the expansion coeff's uniquely:

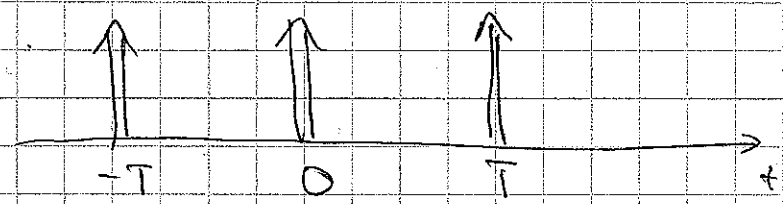
$$\int_a^b \phi_k^*(x)\, f(x)\, dx = \int_a^b \phi_k^*(x) \sum_j a_j \phi_j(x)\, dx$$

$$= a_k$$

Fourier series is an example of this kind of idea!

Example: sequence of delta-functions

$$f(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$



Assert that

$$f(t) = \sum_k a_k e^{j\frac{2\pi}{T}kt}$$

$$\int_{-T/2}^{T/2} e^{-j\frac{2\pi}{T}lt} f(t)\,dt = \int_{-T/2}^{T/2} e^{-j\frac{2\pi}{T}lt} \sum_k a_k e^{j\frac{2\pi}{T}kt}\,dt$$

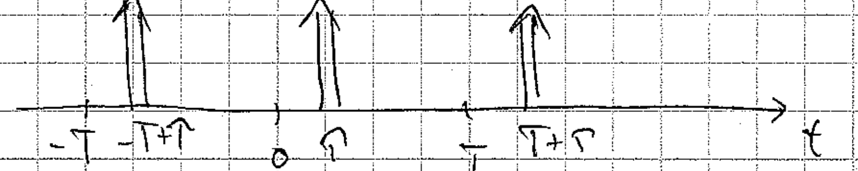
$$= \sum_k a_k \underbrace{\int_{-T/2}^{T/2} e^{j\frac{2\pi}{T}(k-l)t}\,dt}_{=\begin{cases} T & k=l \\ 0 & k\neq l \end{cases}}$$

$$= T a_l$$

$$a_l = \frac{1}{T}\int_{-T/2}^{T/2} e^{-j\frac{2\pi}{T}lt} \sum_{n=-\infty}^{\infty} \delta(t-nT)\,dt$$

$$= \frac{1}{T}$$

Example: Shifted Delta function



$$f(t) = \sum_{n=-\infty}^{\infty} \delta(t - \tau - nT)$$

$$a_\ell = \frac{1}{T} \int_0^T e^{-i\left(\frac{2\pi}{T}\right)\ell t} f(t)\, dt$$

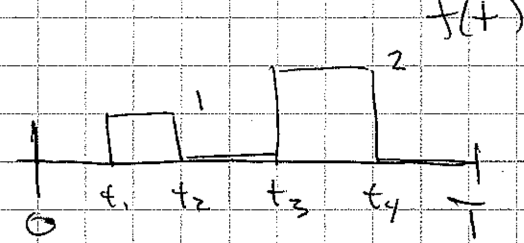
$$= \frac{1}{T} \int_0^T e^{-i\left(\frac{2\pi}{T}\right)\ell t} \delta(t - \tau - nT)\, dt$$

$$= \frac{1}{T} e^{-i\left(\frac{2\pi}{T}\right)\ell \tau}$$

works for $0 < \tau < T$ for this definition, but get same answer even if outside this range

Pick up phase factor associated with a shift in time.

Steps up and down



(Keep in mind Gibbs phenomenon)

$$T a_\ell = \int_0^T e^{-i\left(\frac{2\pi}{T}\right)\ell t} f(t)\, dt$$

$$= \int_{t_1}^{t_2} e^{-i\left(\frac{2\pi}{T}\right)\ell t}\, 1\, dt + \int_{t_3}^{t_4} e^{-i\left(\frac{2\pi}{T}\right)\ell t}\, 2\, dt$$

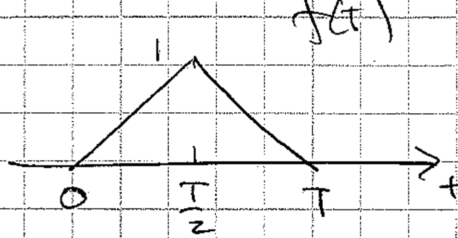
$$= \left.\frac{e^{-i\left(\frac{2\pi}{T}\right)\ell t}}{-i\left(\frac{2\pi}{T}\right)\ell}\right|_{t_1}^{t_2} + 2 \left.\frac{e^{-i\left(\frac{2\pi}{T}\right)\ell t}}{-i\left(\frac{2\pi}{T}\right)\ell}\right|_{t_3}^{t_4}$$

$$= \frac{e^{-i\left(\frac{2\pi}{T}\right)\ell t_2} - e^{-i\left(\frac{2\pi}{T}\right)\ell t_1}}{-i\left(\frac{2\pi}{T}\right)\ell} + 2\, \frac{e^{-i\frac{2\pi}{T}\ell t_4} - e^{-i\left(\frac{2\pi}{T}\right)\ell t_3}}{-i\left(\frac{2\pi}{T}\right)\ell}$$

Thinking about it... Linear superposition

$$T a_\ell = \underbrace{\frac{e^{-i\left(\frac{2\pi}{T}\right)\ell t_1}}{i\left(\frac{2\pi}{T}\right)\ell}}_{\text{step up at } t_1,\text{ height } 1} - \underbrace{\frac{e^{-i\left(\frac{2\pi}{T}\right)\ell t_2}}{i\left(\frac{2\pi}{T}\right)\ell}}_{\text{step down at } t_2,\text{ height } 1} + \underbrace{2\,\frac{e^{-i\left(\frac{2\pi}{T}\right)\ell t_3}}{i\left(\frac{2\pi}{T}\right)\ell}}_{\text{step up at } t_3,\text{ height } 2} - \underbrace{\frac{2e^{-i\left(\frac{2\pi}{T}\right)\ell t_4}}{i\left(\frac{2\pi}{T}\right)\ell}}_{\text{step down at } t_4,\text{ height } 2}$$

Example



$$f(t) = \sum_k a_k e^{j\left(\frac{2\pi}{T}\right)kt}$$

$$a_k = \frac{1}{T}\int_T e^{-j\left(\frac{2\pi}{T}\right)\ell t} f(t)\, dt = \text{a mess}$$

Instead go after $g(t) = \frac{df}{dt}$ $\quad g(t) = \sum_k b_k e^{j\left(\frac{2\pi}{T}\right)kt}$

$$b_\ell = \frac{1}{T}\int_T e^{-j\left(\frac{2\pi}{T}\right)\ell t} g(t)\, dt$$

But, we can relate a_ℓ to b_ℓ :

$$g(t) = \frac{df}{dt}$$

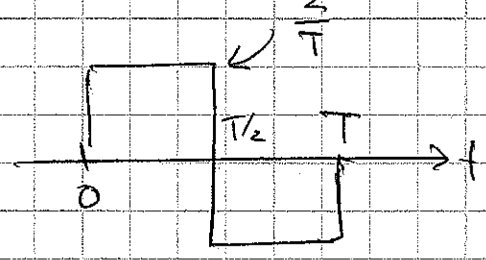
$$\sum_k b_k e^{j\left(\frac{2\pi}{T}\right)kt} = \frac{d}{dt}\sum_k a_k e^{j\left(\frac{2\pi}{T}\right)kt}$$

$$= \sum_k j\left(\frac{2\pi}{T}\right)k\, a_k e^{j\left(\frac{2\pi}{T}\right)kt}$$

So $\quad b_k = j\left(\frac{2\pi}{T}\right) k\, a_k$

$$a_k = \frac{b_k}{j\left(\frac{2\pi}{T}\right)k}$$

Now do $g(t)$



$$T b_\ell = \frac{1}{j\left(\frac{2\pi}{T}\right)\ell} - \frac{2e^{-j\left(\frac{2\pi}{T}\right)\ell\frac{T}{2}}}{j\left(\frac{2\pi}{T}\right)\ell} + \frac{e^{-j\left(\frac{2\pi}{T}\right)\ell T}}{j\left(\frac{2\pi}{T}\right)\ell}$$

Can simplify since $e^{-j\left(\frac{2\pi}{T}\right)\ell\frac{T}{2}} = e^{-j\pi\ell}$

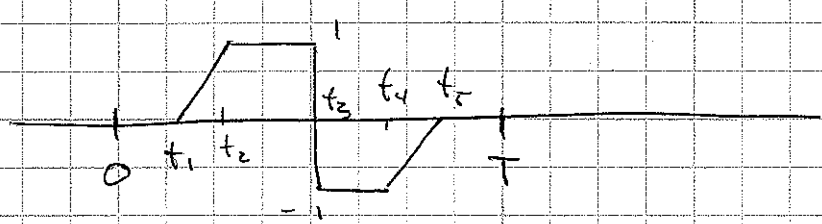
$$e^{-j\left(\frac{2\pi}{T}\right)\ell T} = e^{-j2\pi\ell} = 1$$

$$= \frac{2}{j\left(\frac{2\pi}{T}\right)\ell} - \frac{2e^{-j\pi\ell}}{j\left(\frac{2\pi}{T}\right)\ell}$$

Then

$$T a_\ell = \frac{T b_\ell}{j\left(\frac{2\pi}{T}\right)\ell} = \frac{2}{\left[j\left(\frac{2\pi}{T}\right)\ell\right]^2} - \frac{2e^{-j\pi\ell}}{\left[j\left(\frac{2\pi}{T}\right)\ell\right]^2}$$

Example: more complicated function



$$a_l = \frac{1}{T}\int_0^T e^{-j\left(\frac{2\pi}{T}\right)lt} f(t)\,dt$$

$$= \frac{1}{T}\int_{t_1}^{t_2} e^{-j\left(\frac{2\pi}{T}\right)lt}\,\frac{(t-t_1)}{(t_2-t_1)}\,dt$$

$$+ \frac{1}{T}\int_{t_2}^{t_3} e^{-j\left(\frac{2\pi}{T}\right)lt}\,1\,dt$$

$$+ \frac{1}{T}\int_{t_3}^{t_4} e^{-j\left(\frac{2\pi}{T}\right)lt}\,(-1)\,dt$$

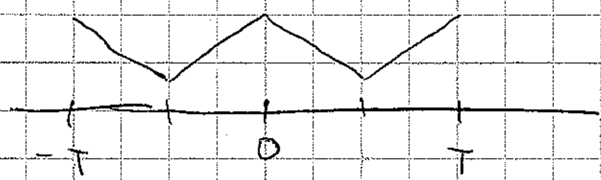
$$+ \frac{1}{T}\int_{t_4}^{t_5} e^{-j\left(\frac{2\pi}{T}\right)}\left[+\frac{t-t_5}{t_5-t_4}\right]dt$$

Then long and messy calculation ...

$$Ta_l = \frac{e^{-j\left(\frac{2\pi}{T}\right)lt_1}}{(t_2-t_1)\left[j\left(\frac{2\pi}{T}\right)l\right]^2} - \frac{e^{-j\left(\frac{2\pi}{T}\right)lt_2}}{(t_2-t_1)\left[j\left(\frac{2\pi}{T}\right)l\right]^2} - 2\,\frac{e^{-j\left(\frac{2\pi}{T}\right)lt_3}}{j\left(\frac{2\pi}{T}\right)l}$$

$$+ \frac{e^{-j\left(\frac{2\pi}{T}\right)lt_4}}{(t_5-t_4)\left[j\left(\frac{2\pi}{T}\right)l\right]^2} - \frac{e^{-j\left(\frac{2\pi}{T}\right)lt_5}}{(t_5-t_4)\left[j\left(\frac{2\pi}{T}\right)l\right]^2}$$

Example: $f(t)$ is even



$$f(t) = f(-t)$$

What can we say about a_ℓ?

$$a_\ell = \frac{1}{T}\int_{-T/2}^{T/2} e^{-j\left(\frac{2\pi}{T}\right)\ell t} f(t)\, dt$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} e^{-j\left(\frac{2\pi}{T}\right)\ell t} f(-t)\, dt$$

let $t' = -t$

$dt' = -dt$

$$= -\frac{1}{T}\int_{T/2}^{-T/2} e^{j\left(\frac{2\pi}{T}\right)\ell t'} f(t')\, dt'$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} e^{j\left(\frac{2\pi}{T}\right)\ell t'} f(t')\, dt'$$

$$= a_{-\ell}$$

if $f(t)$ is even, then $a_\ell = a_{-\ell}$