6.003 R

April 2, 2010

Integral of products

Define $f(t) = \sum_k a_k e^{j\left(\frac{2\pi}{T}\right)kt}$

$$g(t) = \sum_k b_k e^{j\left(\frac{2\pi}{T}\right)kt}$$

Then

$$\int_T g^*(t) f(t)\, dt$$

$$= \int_T \left[\sum_k b_k^* e^{-j\left(\frac{2\pi}{T}\right)kt}\right]\left[\sum_{k'} a_{k'} e^{j\left(\frac{2\pi}{T}\right)k't}\right] dt$$

$$= \sum_k \sum_l b_k^* a_l \underbrace{\int_T e^{j\left(\frac{2\pi}{T}\right)(l-k)t}\, dt}_{= \begin{cases} T & l = k \\ 0 & l \neq k \end{cases}}$$

$$= T \sum_k b_k^* a_k$$

As a result, we may write

$$\int_T |f(t)|^2 dt = T \sum_k |a_k|^2$$

Product formula

$$f(t) = \sum_k a_k e^{j\left(\frac{2\pi}{T}\right)kt}$$

$$g(t) = \sum_k b_k e^{j\left(\frac{2\pi}{T}\right)kt}$$

$$f(t)g(t) = \left[\sum_k a_k e^{j\left(\frac{2\pi}{T}\right)kt}\right]\left[\sum_\ell b_\ell e^{j\left(\frac{2\pi}{T}\right)\ell t}\right]$$

$$= \sum_{k\ell} a_k b_\ell \, e^{j\left(\frac{2\pi}{T}\right)(k+\ell)t}$$

let $m = k+\ell \Rightarrow \ell = m-k$

$$f(t)\,g(t) = \sum_k \sum_m a_k b_{m-k} \, e^{j\frac{2\pi}{T}mt}$$

Define $c_m = \sum_k a_k b_{m-k}$ $\qquad c_m = (a * b)_m$

$$f(t)\,g(t) = \sum_{m} c_m e^{j\left(\frac{2\pi}{T}\right)mt}$$

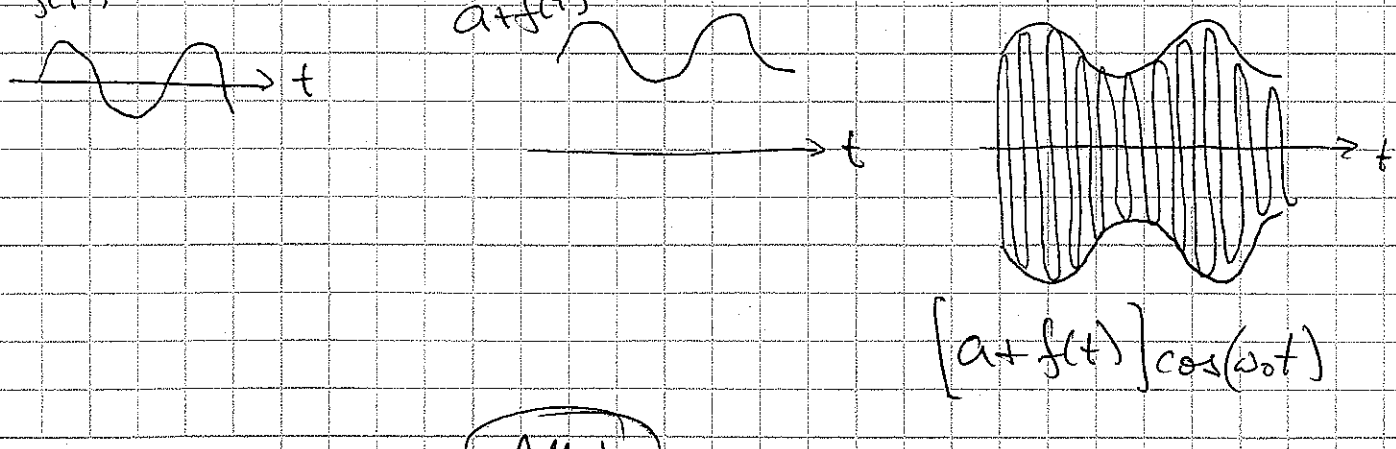
AM Radio Receiver

Radio transmitter　　　　Receiver

Later on this term, we will talk more about Radio. Here we are interested in a simple demodulation scheme, which motivates us to be interested in filters.

Consider AM Radio (AM = Amplitude modulation)

Start with audio　　add an offset　　then modulate

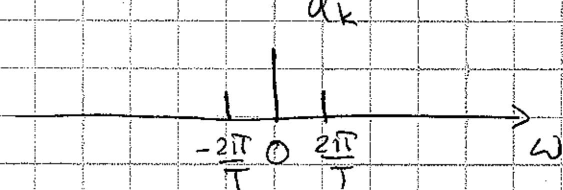


$$[a + f(t)] \cos(\omega_0 t)$$

Signals in frequency

$$a + f(t) = \sum_k a_k e^{j\left(\frac{2\pi}{T}\right)kt}$$

for $f(t) = \cos\left(\frac{2\pi}{T}t\right)$, we get (as an example of an audio signal)



for $\left[a + f(t)\right]\cos\omega_0 t$ (modulated signal) we get

$$\left[a + f(t)\right]\cos\omega_0 t = \sum_k a_k e^{j\left(\frac{2\pi}{T}\right)kt}\left[\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}\right]$$

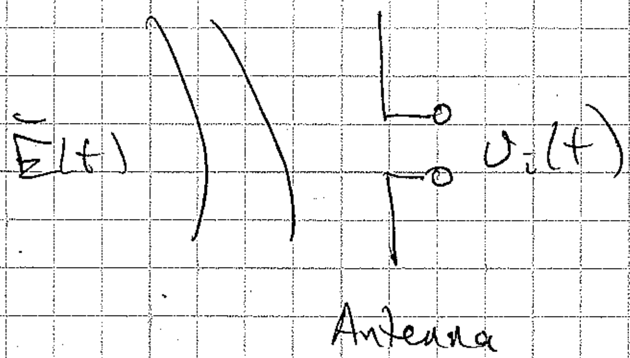
$$= \frac{1}{2}\sum_k a_k\left[e^{j\left[\left(\frac{2\pi}{T}\right)k + \omega_0\right]t} + e^{j\left[\left(\frac{2\pi}{T}\right)k - \omega_0\right]t}\right]$$

$\frac{1}{2}a_k$ shifted by $-\omega_0$

$\frac{1}{2}a_k$ shifted by ω_0

What would happen if you drove a speaker with this kind of signal?

Receiver

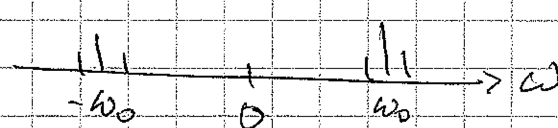


Antenna sees local electric field, voltage develops

$$v_i(t) \sim \left[a + f(t)\right] \cos(\omega_0 t + \phi)$$

phase shift that we will ignore in what follows

This voltage has high frequency components



Audio frequencies ~ 100s to 1000s Hz

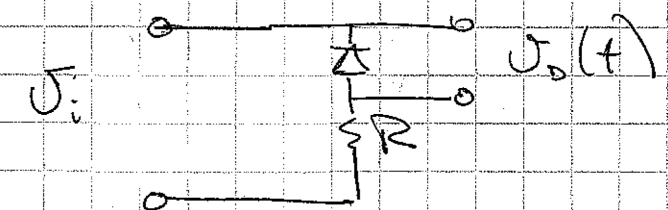
RF frequencies ~ 10^6 Hz

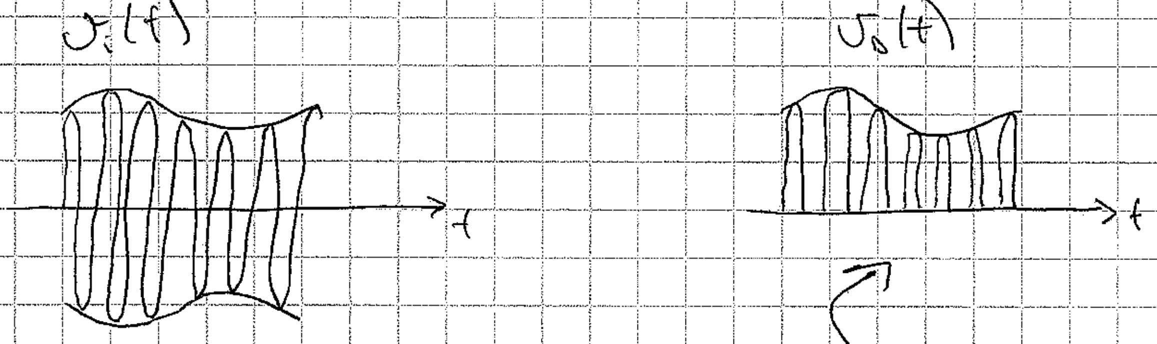
Need to demodulate before we get to hear audio

A demodulation scheme

there are different ways to recover the audio signal (which is called demodulation).

One approach is to begin using rectification.





Negative part of signal missing & $v_i(t)$ is rectified to give $v_o(t)$.

This circuit is not an LTI system.

So, what frequencies present in rectified signal?

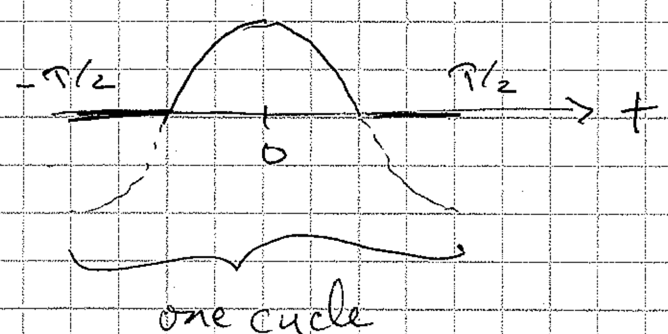
$$v_D(t) \sim \left[a + f(t)\right] \cos(\omega_0 t)\; u\left(\cos \omega_0 t\right)$$

Start with rectified RF piece:

$$\cos(\omega_0 t)\; u\left(\cos\omega_0 t\right) = \sum_k b_k e^{j\omega_0 k t}$$

$$b_k = \frac{1}{T} \int_T e^{-j\omega_0 k t} \cos(\omega_0 t)\, u\left(\cos\omega_0 t\right) dt$$

$$\omega_0 T = 2\pi \qquad T = \frac{2\pi}{\omega_0}$$



$$b_k = \frac{1}{T} \int_{-T/4}^{T/4} e^{-j\omega_0 k t} \cos(\omega_0 t)\, dt$$

AM-5

$$= \frac{1}{T}\int_{-T/4}^{T/4} e^{-j\omega_0 k t}\left[\frac{e^{j\omega_0 t}+e^{-j\omega_0 t}}{2}\right]dt$$

$$= \frac{1}{T}\int_{-T/4}^{T/4} \frac{e^{-j\omega_0(k-1)t}+e^{-j\omega_0(k+1)t}}{2}\,dt$$

$$= \frac{1}{2T}\left[\left.\frac{e^{-j\omega_0(k-1)t}}{-j\omega_0(k-1)}\right|_{-T/4}^{T/4} + \left.\frac{e^{-j\omega_0(k+1)t}}{-j\omega_0(k+1)}\right|_{-T/4}^{T/4}\right]$$

$$= \frac{1}{2T}\left[\frac{e^{-j\omega_0(k-1)\frac{T}{4}}-e^{+j\omega_0(k-1)\frac{T}{4}}}{-j\omega_0(k-1)} + \frac{e^{-j\omega_0(k+1)\frac{T}{4}}-e^{+j\omega_0(k+1)\frac{T}{4}}}{-j\omega_0(k+1)}\right]$$

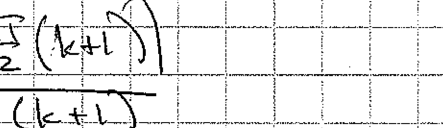
$\frac{T}{4} = \frac{\pi}{2\omega_0}$

$$= \frac{\sin\left[\frac{\pi}{2}(k-1)\right]}{\omega_0 T\,(k-1)} + \frac{\sin\left[\frac{\pi}{2}(k+1)\right]}{\omega_0 T\,(k+1)}$$

$\omega_0 T = 2\pi$

$$= \frac{1}{2\pi}\left[\frac{\sin\frac{\pi}{2}(k-1)}{(k-1)} + \frac{\sin\left[\frac{\pi}{2}(k+1)\right]}{(k+1)}\right]$$

$k = 0 \qquad \frac{1}{2\pi}\left[\frac{\sin(-\frac{\pi}{2})}{-1} + \frac{\sin\frac{\pi}{2}}{1}\right] = \frac{1}{\pi}$



$k = 1 \qquad \frac{1}{2\pi}\left[\cancel{\frac{\sin 0\frac{\pi}{2}}{0}} + \frac{\sin\frac{\pi}{2}.2}{2}\right] = \frac{1}{4}$

$$\lim_{k-1\to 0}\frac{\sin\frac{\pi}{2}(k-1)}{(k-1)} \to \frac{\pi}{2}$$

$k = 2 \qquad \frac{1}{2\pi}\left[\frac{\sin\frac{\pi}{2}}{1} + \frac{\sin\frac{3\pi}{2}}{3}\right] = \frac{1}{2\pi}\left[1 - \frac{1}{3}\right] = \frac{1}{3\pi}$

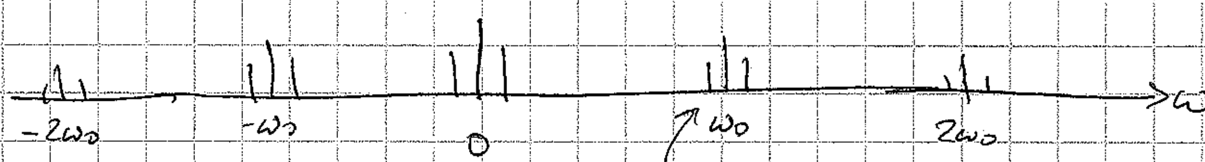
Does this help?

$$v_D(t) \sim \left[a + f(t)\right] \underbrace{\cos(\omega_0 t)\, u(\cos\omega_0 t)}_{\sum_k b_k e^{j\omega_0 k t}}$$

$$\left[a + f(t)\right] \to \sum_k a_k e^{j\left(\frac{2\pi}{T}\right)kt} \qquad \sum_k b_k e^{j\omega_0 k t}$$

Resulting frequency distribution is going to be a convolution (also periodic if ω_0 integer multiple of $\frac{2\pi}{T}$)

$$\sim \sum_k a_k e^{j\left(\frac{2\pi}{T}\right)kt}$$

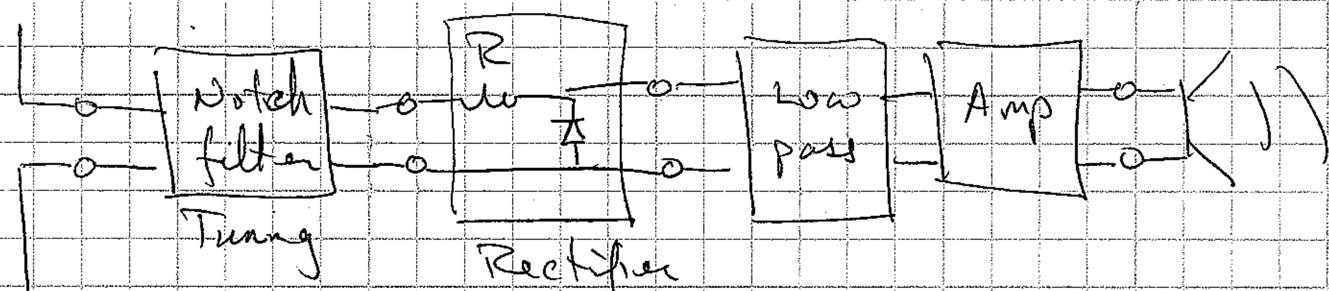


$$\sim \sum_k a_k e^{j\left(\frac{2\pi}{T}\right)kt} \times e^{j\omega_0 t}$$

It looks like the diode output has frequency components centered around $\omega = 0$, so that a version of the audio signal is contained.

If we drove a speaker with this signal, then we could hear the audio faintly (would want an amplifier). Speaker doesn't respond well for RF signals

There are many RF stations, so need to augment ~~design~~ design to have a useful Radio.



This kind of design would do a better job.