6.003 R

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Notation

Laplace transform:

$$\underline{X}(s) = \int_{-\infty}^{\infty} e^{-st} x(t)\, dt$$

$$\underline{X}(j\omega) = \int_{-\infty}^{\infty} e^{-j\omega t} x(t)\, dt$$

Fourier transform written using Laplace transform notation

Fourier transform notation

$$\underline{X}(\omega) = \int_{-\infty}^{\infty} e^{-j\omega t} x(t)\, dt$$

this is used in other classes, books, and papers.

Example: Integral of a product

Let

$$f(t) = \int_{-\infty}^{\infty} F(j\omega)\, e^{j\omega t}\, \frac{d\omega}{2\pi}$$

$$g(t) = \int_{-\infty}^{\infty} G(j\omega)\, e^{j\omega t}\, \frac{d\omega}{2\pi}$$

$$\int_{-\infty}^{\infty} g^*(t)\, f(t)\, dt =$$

$$\int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} G(j\omega)\, e^{j\omega t}\, \frac{d\omega}{2\pi}\right]^* \left[\int_{-\infty}^{\infty} F(j\omega)\, e^{j\omega t}\, \frac{d\omega}{2\pi}\right] dt$$

$$= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi}\, G^*(j\omega')\, F(j\omega) \int_{-\infty}^{\infty} e^{j(\omega-\omega')t}\, dt \cancel{d\omega}$$

$$\int_{-\infty}^{\infty} e^{j(\omega-\omega')t}\, dt = 2\pi\, \delta(\omega-\omega')$$

$$\left[\text{Check} \quad \int_{-\infty}^{\infty} e^{-j\omega t}\, 2\pi\, \delta(\omega-\omega')\, \frac{d\omega}{2\pi} = e^{-j\omega' t} \right]$$

$$= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi}\, G^*(j\omega')\, F(j\omega)\, 2\pi\, \delta(\omega-\omega')$$

$$= \int_{-\infty}^{\infty} G^*(j\omega)\, F(j\omega)\, \frac{d\omega}{2\pi}$$

Product formula

$$f(t) = \int_{-\infty}^{\infty} F(j\omega)\, e^{j\omega t} \frac{d\omega}{2\pi}$$

$$g(t) = \int_{-\infty}^{\infty} G(j\omega)\, e^{j\omega t} \frac{d\omega}{2\pi}$$

$$h(t) = f(t)\, g(t) = \int_{-\infty} H(j\omega)\, e^{j\omega t} \frac{d\omega}{2\pi}$$

$$H(j\omega) = \int_{-\infty}^{\infty} e^{-j\omega t} f(t)\, g(t)\, dt$$

$$= \int_{-\infty}^{\infty} e^{-j\omega t} \left[\int_{-\infty}^{\infty} F(j\omega')\, e^{j\omega' t} \frac{d\omega'}{2\pi}\right]\left[\int_{-\infty}^{\infty} G(j\omega'')\, e^{j\omega'' t} \frac{d\omega'}{2\pi}\right] dt$$

$$= \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega''}{2\pi}\, F(j\omega')\, G(j\omega'') \underbrace{\int_{-\infty}^{\infty} e^{-j\omega t} e^{j\omega' t} e^{j\omega'' t}\, dt}_{2\pi\, \delta(\omega - \omega' - \omega'')}$$

$$= \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega''}{2\pi}\, F(j\omega')\, G(j\omega'')\, 2\pi\, \delta(\omega - \omega' - \omega'')$$

$$= \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi}\, F(j\omega')\, G(j(\omega - \omega'))$$

$$= \frac{1}{2\pi} (F * G)(j\omega)$$

Product formula going other way

$$F(j\omega) = \int_{-\infty}^{\infty} e^{-j\omega t} f(t)\, dt$$

$$G(j\omega) = \int_{-\infty}^{\infty} e^{-j\omega t} g(t)\, dt$$

$$F(j\omega)\, G(j\omega) = \int_{-\infty}^{\infty} e^{-j\omega t} h(t)\, dt$$

$$h(t) = \int_{-\infty}^{\infty} F(j\omega)\, G(j\omega)\, e^{j\omega t} \frac{d\omega}{2\pi}$$

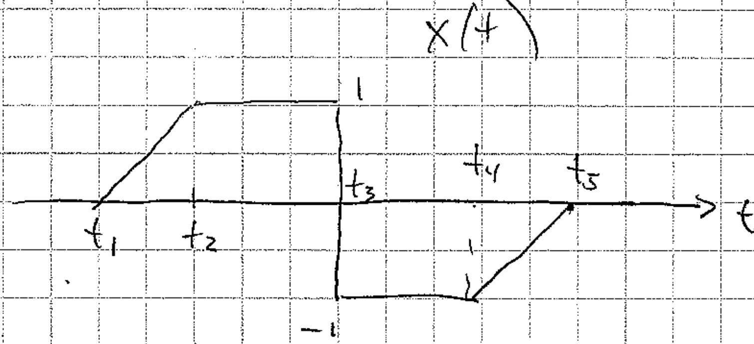
$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} e^{-j\omega t'} f(t')\, dt'\right]\left[\int_{-\infty}^{\infty} e^{-j\omega t''} g(t'')\, dt''\right] e^{j\omega t} \frac{d\omega}{2\pi}$$

$$= \int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} dt''\, f(t')\, g(t'') \underbrace{\int_{-\infty}^{\infty} e^{-j\omega t'} e^{-j\omega t''} e^{j\omega t} \frac{d\omega}{2\pi}}_{\delta(t-t''-t'')}$$

$$= \int_{-\infty}^{\infty} dt'\, f(t')\, g(t-t')$$

$$= (f * g)(t)$$

Fourier transform and superposition



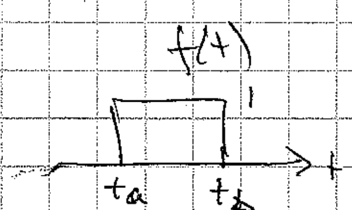
$$X(j\omega) = \int_{-\infty}^{\infty} e^{-j\omega t} x(t)\, dt$$

$$= \int_{t_1}^{t_2} e^{-j\omega t} \frac{t - t_1}{t_2 - t_1}\, dt + \int_{t_2}^{t_3} e^{-j\omega t}\, 1\, dt$$

$$+ \int_{t_3}^{t_4} e^{-j\omega t} (-1)\, dt + \int_{t_4}^{t_5} e^{-j\omega t} \frac{t - t_5}{t_5 - t_4}\, dt$$

Should be able to use linear superposition

Need to understand offsets, steps and ramps



$$F(j\omega) = \int_{-\infty}^{\infty} e^{-j\omega t} f(t)\, dt = \int_{t_a}^{t_b} e^{-j\omega t} (1)\, dt$$

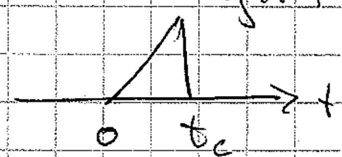
$$= \left.\frac{e^{-j\omega t}}{-j\omega}\right|_{t_a}^{t_b} = \frac{e^{-j\omega t_b}}{-j\omega} - \frac{e^{-j\omega t_a}}{-j\omega}$$

So step up at t_a looks like

$$\frac{e^{-j\omega t_a}}{j\omega}$$

OK - what about a ramp?

$$g(t) = t \quad \text{for} \quad 0 < t < t_c$$



$$G(j\omega) = \int_{-\infty}^{\infty} e^{-j\omega t}\, g(t)\, dt$$

$$= \int_0^{t_c} e^{-j\omega t}\, t\, dt$$

$$= j \frac{d}{d\omega} \int_0^{t_c} e^{-j\omega t}\, dt$$

$$= +j \frac{d}{d\omega} \left[\frac{e^{-j\omega t}}{-j\omega} \Big|_0^{t_c} \right]$$

$$= j \frac{d}{d\omega} \left[\frac{e^{-j\omega t_c} - 1}{-j\omega} \right]$$

$$= j \left[\frac{-j t_c e^{-j\omega t_c}}{-j\omega} - \frac{(e^{-j\omega t_c} - 1)}{-j\omega^2} \right]$$

$$= \frac{1}{(j\omega)^2} - \frac{e^{-j\omega t_c}}{(j\omega)^2} - \frac{e^{-j\omega t_c}}{j\omega}$$

(up) Ramp at $t=0$ | down Ramp at $t=t_c$ | step down at $t=t_c$

OK looks like it works OK. So for original example, we may write

$$\bar{X}(j\omega) = \frac{e^{-j\omega t_1}}{(t_2-t_1)(j\omega)^2} - \frac{e^{-j\omega t_2}}{(t_2-t_1)(j\omega)^2} - \frac{2e^{-j\omega t_3}}{j\omega} + \frac{e^{-j\omega t_4}}{(j\omega)^2(t_5-t_4)} - \frac{e^{-j\omega t_5}}{(j\omega)^2(t_5-t_4)}$$

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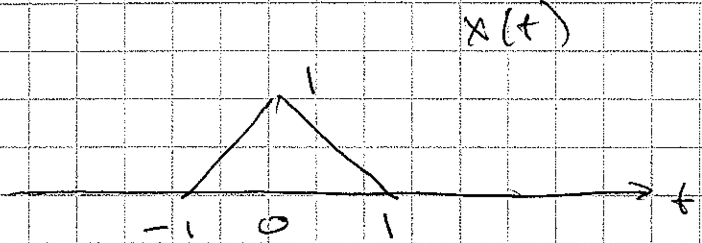
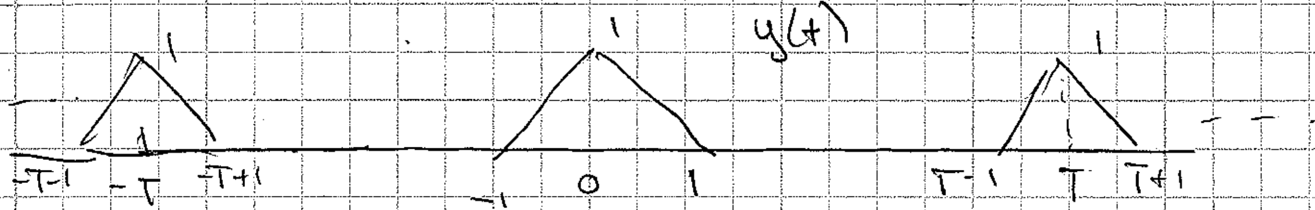
Compare





Interested in:

(a) $X(j\omega)$

(b) $Y(j\omega)$

(c) a_k for $y(t) = \sum_k a_k e^{j\left(\frac{2\pi}{T}\right)kt}$

(a) $$X(j\omega) = \frac{e^{+j\omega}}{(j\omega)^2} - \frac{2}{(j\omega)^2} + \frac{e^{-j\omega}}{(j\omega)^2}$$

Do (c) next ...

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} e^{-j\frac{2\pi}{T}kt}\, y(t)\, dt$$

$$T a_k = \frac{e^{j\omega}}{\left(j\frac{2\pi}{T}k\right)^2} - \frac{2}{\left(j\frac{2\pi}{T}k\right)^2} + \frac{e^{-j\omega}}{\left(j\frac{2\pi}{T}k\right)^2}$$

Next do (b)

$$y(t) = x(t) * \sum_n \delta(t-nT)$$

What is transform of a series of δ-functions? Also a series of δ-functions. Recall argument from class. First do it as a series

$$\sum_n \delta(t-nT) = \sum_k b_k e^{j\frac{2\pi}{T}kt}$$

$$b_k = \frac{1}{T}\int_{-T/2}^{T/2} e^{-j\frac{2\pi}{T}kt}\,\delta(t)\,dt$$

$$= \frac{1}{T}$$

$$\sum_n \delta(t-nT) = \sum_k \frac{1}{T} e^{j\frac{2\pi}{T}kt}$$

So, what $B(j\omega)$ gives

$$9 \qquad \int_{-\infty}^{\infty} B(j\omega) e^{j\omega t}\frac{d\omega}{2\pi} = \sum_k \frac{1}{T} e^{j\frac{2\pi}{T}kt}$$

$$B(j\omega) = \sum_k \frac{2\pi}{T}\,\delta\!\left(\omega - \frac{2\pi}{T}k\right)$$

$$\sum_n \delta(t-nT) = \int_{-\infty}^{\infty} B(j\omega)\, e^{j\omega t}\,\frac{d\omega}{2\pi}$$

then

$$x(t) * \sum_n \delta(t-nT) \longleftrightarrow X(j\omega)B(j\omega)$$

$$\text{So}\quad X(j\omega)B(j\omega) = \left[\frac{e^{j\omega} - 2 + e^{-j\omega}}{(j\omega)^2}\right]\sum_k \frac{2\pi}{T}\,\delta\!\left(\omega - \frac{2\pi}{T}k\right)$$

$$= \sum_k \frac{2\pi}{T}\,\frac{e^{j\frac{2\pi}{T}k} - 2 + e^{-j\frac{2\pi}{T}k}}{\left(j\frac{2\pi}{T}k\right)^2}\,\delta\!\left(\omega - \frac{2\pi k}{T}\right)$$

Compare with series, and conclude that

$$\sum_k 2\pi a_k\,\delta\!\left(\omega - \frac{2\pi}{T}k\right)$$