6.003 R

April 14, 2010.

Peter Hagelstein

Discrete time Fourier Transform

$$x[n] = \int_{2\pi} X(e^{j\Omega}) e^{j\Omega n} \frac{d\Omega}{2\pi}$$

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} e^{-j\Omega n} x[n]$$

Can we show that it works?

$$x[n] = \int_{2\pi} X(e^{j\Omega}) e^{j\Omega n} \frac{d\Omega}{2\pi}$$

$$= \int_{2\pi} \left[\sum_{m=-\infty}^{\infty} e^{-j\Omega m} x[m] \right] e^{j\Omega n} \frac{d\Omega}{2\pi}$$

$$= \sum_{m=-\infty}^{\infty} x[m] \underbrace{\int_{2\pi} e^{j\Omega(n-m)} \frac{d\Omega}{2\pi}}_{\begin{array}{ll} = 1 & n=m \\ = 0 & n \neq m \end{array}}$$

$$= \sum_{m=-\infty}^{\infty} x[m] \, \delta[n-m]$$

$$= x[n]$$

It works!

Example

$$x[n] = \delta[n]$$

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} e^{-j\Omega n} x[n]$$

$$= 1$$

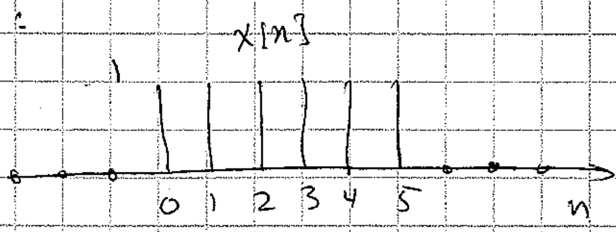
Example: shift

$$x[n] = \delta[n-N]$$

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} e^{-j\Omega n} \delta[n-N]$$

$$= e^{-j\Omega N}$$

Example:



$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} e^{-j\Omega n} x[n]$$

$$= \sum_{k=0}^{5} e^{-j\Omega k}$$

Can sum it like a geometric series

$$\sum_{k=0}^{5} \alpha^k e^{-j\Omega k} = \sum_{k=0}^{\infty} \alpha^k e^{-j\Omega k} - \sum_{k=6}^{\infty} \alpha^k e^{-j\Omega k}$$

$$= \frac{1}{1 - \alpha e^{-j\Omega}} - \frac{\alpha^6 e^{-j\Omega 6}}{1 - \alpha e^{-j\Omega}}$$

$$= \frac{1 - \alpha^6 e^{-j\Omega 6}}{1 - \alpha e^{-j\Omega}}$$

Then take limit $\alpha \to 1$

$$X(e^{j\Omega}) = \frac{1 - e^{-j6\Omega}}{1 - e^{-j\Omega}}$$

Singularity near $\Omega = 0$

$$\to \frac{1 - (1 - j6\Omega + \cdots)}{1 - (1 - j\Omega + \cdots)} = \frac{j6\Omega + \cdots}{j\Omega + \cdots} \to 6$$

So, it has the right limit

Think about a Ramp

Start with

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} e^{-j\Omega n} x[n]$$

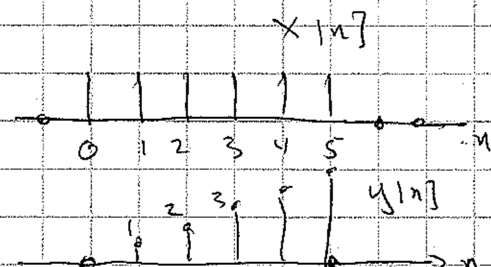
$$Y(e^{j\Omega}) = j\frac{d}{d\Omega} X(e^{j\Omega}) = j\frac{d}{d\Omega}\sum_{n=-\infty}^{\infty} e^{-j\Omega n} x[n]$$

$$= +j \sum_{n=-\infty}^{\infty} (-jn) e^{-j\Omega n} x[n]$$

$$\boxed{Y(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} e^{-j\Omega n}\, n\, x[n]} \Rightarrow y[n] = n\,x[n]$$

OK, Consider example we just did

$$X(e^{j\Omega}) = \frac{1 - e^{-j6\Omega}}{1 - e^{-j\Omega}}$$



$$Y(e^{j\Omega}) = j\frac{d}{d\Omega} X(e^{j\Omega})$$

$$= j\frac{d}{d\Omega}\left(\frac{1 - e^{-j6\Omega}}{1 - e^{-j\Omega}}\right)$$

$$= -j\,\frac{(1 - e^{-j6\Omega})(je^{-j\Omega})}{(1 - e^{-j\Omega})^2} - \frac{6e^{-j6\Omega}}{1 - e^{-j\Omega}}$$

$$Y(e^{j\Omega}) = \frac{e^{-j\Omega}}{(1-e^{-j\Omega})^2} - \frac{e^{-j7\Omega}}{(1-e^{-j\Omega})^2} - 6\frac{e^{-j6\Omega}}{1-e^{-j\Omega}}$$

Thinking about linear superposition...

Recall Z-transform

$$n\,u[n] \longleftrightarrow \frac{e^{-j\Omega}}{(1-e^{-j\Omega})^2} \qquad \frac{z}{(1-z)^2}$$

$$(n+6)\,u[n+6] \longleftrightarrow e^{-j6\Omega}\,\frac{e^{-j\Omega}}{(1-e^{-j\Omega})^2} \qquad \text{using } z = e^{+j\Omega}$$

$$u[n+6] \longleftrightarrow e^{-j6\Omega}\,\frac{1}{1-e^{-j\Omega}}$$

Ramp up starting at $n=0$

Ramp straightening out at $n=6$

then step down by 6 at $n=6$

Z-transform of ramp

$$x[n] = n u[n]$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

Before we had argued that $\frac{z}{z-1}$ acts as an accumulator so that

$$X(z) = \frac{z}{z-1} \sum_{n=-\infty}^{\infty} u[n-1] z^{-n}$$

$$= \frac{z}{(z-1)^2}$$

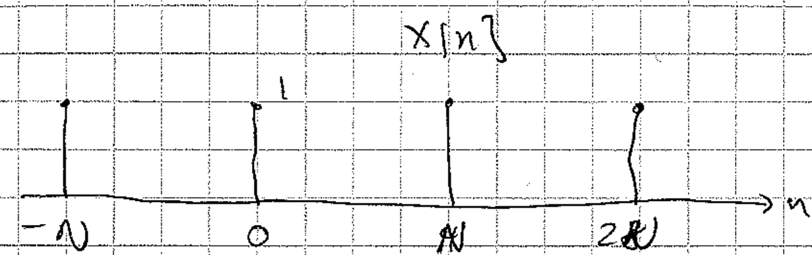
Connection between ~~z~~ Z-transform + Fourier transform

$$X(e^{j\Omega}) = X(z)\Big|_{z=e^{j\Omega}} = \frac{e^{j\Omega}}{(e^{j\Omega}-1)^2}$$

$$= \frac{e^{-j\Omega}}{(1-e^{-j\Omega})^2}$$

in agreement with our Fourier transform argument

What about periodic signal?



Want Fourier transform $X(e^{j\Omega})$

In this case, start with Fourier series

$$x[n] = \sum_{k=0}^{N-1} a_k e^{j\frac{2\pi}{N}kn}$$

$$a_k = \cancel{\frac{1}{N}\sum_{k=0}^{N-1} e^{-j\frac{2\pi}{N}kn} x[n]}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}kn} x[n]$$

So here

$$a_k = \frac{1}{N}\sum_{n=0}^{N-1} \cancel{e^{j\frac{2\pi}{N}kn}}\ \delta[n]\, e^{-j\frac{2\pi}{N}kn}$$

$$= \frac{1}{N}$$

$$x[n] = \sum_{k=0}^{N-1} \frac{1}{N} e^{j\frac{2\pi}{N}kn} \rightarrow$$

$n=0$ get 1

$n\neq 0$ cancellation, get 0

Now we want Fourier transform

$$\text{Try} \quad X(e^{j\Omega}) = \sum_{k=0}^{N-1} b_k \, \delta\left(\Omega - \frac{2\pi}{N}k\right)$$

Now, see if we get same $x[n]$

$$x[n] = \int_{2\pi} X(e^{j\Omega}) \, e^{j\Omega n} \frac{d\Omega}{2\pi}$$

$$= \int_{2\pi} \left[\sum_{k=0}^{N-1} b_k \, \delta\left(\Omega - \frac{2\pi}{N}k\right) \right] e^{j\Omega n} \frac{d\Omega}{2\pi}$$

$$= \sum_{k=0}^{N-1} b_k \int_{2\pi} \delta\left(\Omega - \frac{2\pi k}{N}\right) e^{j\Omega n} \frac{d\Omega}{2\pi}$$

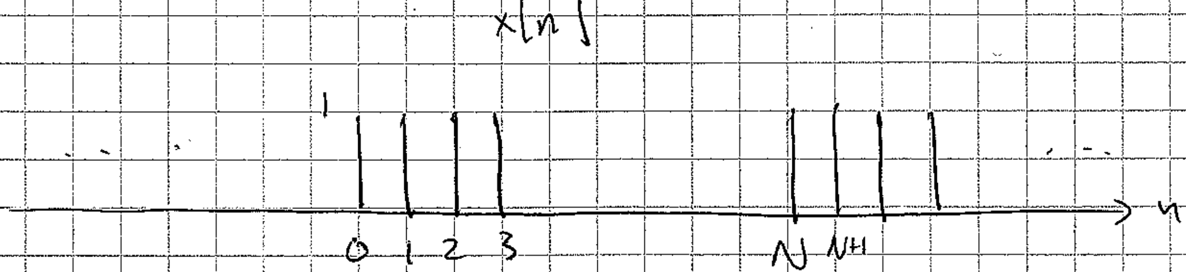
$$= \sum_{k=0}^{N-1} \frac{b_k}{2\pi} \, e^{j\frac{2\pi}{N}kn}$$

It works if $\quad \dfrac{b_k}{2\pi} = a_k$

So for this example we get

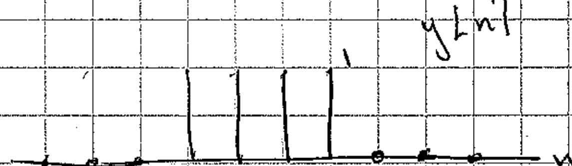
$$X(e^{j\Omega}) = \frac{2\pi}{N} \sum_{k=0}^{N-1} \cancel{\frac{1}{N}} \, \cancel{e^{j\frac{2\pi}{N}kn}} \, \delta\left(\Omega - \frac{2\pi}{N}k\right)$$

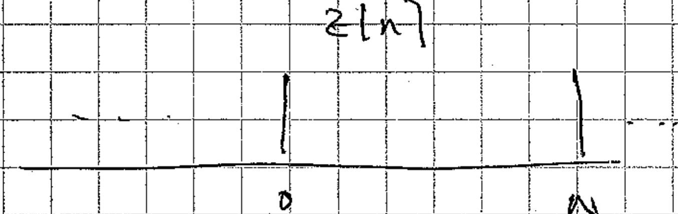
Example



Want Fourier transform ~~of~~ $X(e^{j\Omega})$

Think about it as convolution





We would expect $x[n] = (y * z)[n]$ to be consistent with $X(e^{j\Omega}) = Y(e^{j\Omega})Z(e^{j\Omega})$

Can we show this?

$$x[n] \overset{?}{=} \int_{2\pi} Y(e^{j\Omega})\, Z(e^{j\Omega})\, e^{j\Omega n} \frac{d\Omega}{2\pi}$$

$$= \int_{2\pi} \Big[\sum_k y[k] e^{-j\Omega k} \Big] \Big[\sum_\ell z[\ell] e^{-j\Omega \ell} \Big] e^{j\Omega n} \frac{d\Omega}{2\pi}$$

$$= \sum_k \sum_\ell y[k] z[\ell] \underbrace{\int_{2\pi} e^{j\Omega(n-k-\ell)} \frac{d\Omega}{2\pi}}_{\delta(n-k-\ell)}$$

$$= \sum_k y[k] z[n-k] = (y * z)[n]$$

$$Y(e^{j\Omega}) = \frac{1 - e^{-5j\Omega}}{1 - e^{-j\Omega}}$$

$$Z(e^{j\Omega}) = \frac{2\pi}{N} \sum_{k=0}^{N-1} \delta\left(\Omega - \frac{2\pi}{N}k\right)$$

$$X(e^{j\Omega}) = Y(e^{j\Omega})\, Z(e^{j\Omega})$$

$$= \frac{2\pi}{N} \sum_{k=0}^{N-1} \delta\left(\Omega - \frac{2\pi}{N}k\right) \frac{1 - e^{-5j\frac{2\pi}{N}k}}{1 - e^{-j\frac{2\pi}{N}k}}$$