6.003 Rec. | Wednesday Apr. 14, Sections 3 & 4

Today:

1. Analysis & synthesis of DT fourier series
2. Geometric picture for period 4 & period 8 synthesis
3. The fast fourier transform (FFT)
4. Algorithmic scaling of FFT vs. convolution

(1) $x[n-N] = x[n]$ ← periodic $\qquad x[n] = \sum_{k=0}^{N-1} a_k e^{j\frac{2\pi kn}{N}}$ "synthesis"

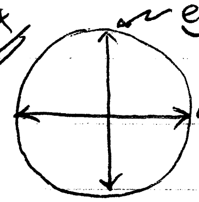
$$a_k = \frac{1}{N}\sum_{\text{period}} x[n]\, e^{-j\frac{2\pi k}{N} n} \qquad \text{"analysis"}$$

Both equations give similar matrix equations. $N=4$, In matrix form,

$$\begin{bmatrix} x[0] \\ \vdots \\ x[N-1] \end{bmatrix} = \begin{bmatrix} e^{j\frac{\pi 2}{N}\cdot 0 \cdot 0} & \cdots & e^{j\frac{2\pi}{N}(N-1)\cdot 0} \\ & \ddots & \\ e^{j\frac{2\pi}{N}0\cdot N-1} & & \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{N-1} \end{bmatrix} \quad \leftarrow \text{synthesis.}$$

We can write a similar matrix transformation for analysis, i.e. if we can do one matrix computation efficiently, then we can do the other. (The matrices are inverses from eachother.)

(2) period 4

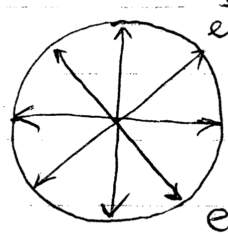


$e^{j\frac{2\pi}{4}\cdot 1\cdot 1}$ first harmonic

$e^{j\frac{2\pi}{4}\cdot 0\cdot 1}$, where $n=1$ (a time snap shot)

$e^{j\frac{2\pi}{4}\cdot 3\cdot 1}$

period 8



$e^{j\frac{2\pi}{8}\cdot 1\cdot 1}$

Shown for $n=1$

$e^{j\frac{2\pi}{8}\cdot 0\cdot 1}$

$e^{j\frac{2\pi}{8}7\cdot 1}$

3 Period 8 Case (synthesis)

$$x[n] = X[0]e^{j2\pi\frac{0\cdot n}{8}} + X[2]e^{j\frac{2\pi\,2\cdot n}{8}} + X[4]e^{j\frac{2\pi\cdot 4\cdot n}{8}} + X[6]e^{j\frac{2\pi}{8}\cdot 6\cdot n} \Big\} x_{even}$$

$$+$$

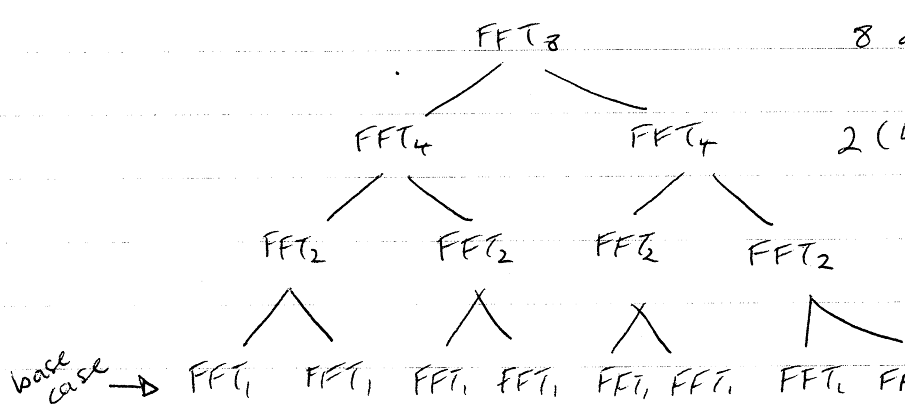
$$X[1]e^{j\frac{2\pi\cdot 1n}{8}} + X[3]e^{j\,2\pi 3/8\, n} + X[5]e^{j\frac{2\pi\,5n}{8}} + X[7]e^{j\frac{2\pi\,7n}{8}}$$

$$= x_{even}[n] + e^{j\frac{2\pi}{8}n\cdot 1}\, x_{odd}[n],$$

where $x_{odd}[n] = X[1]e^{j\frac{2\pi\,0}{8}} + X[3]e^{j\frac{2\pi\,2\cdot n}{8}} + \cdots + X[7]e^{j\frac{2\pi\,6n}{8}}$.

$x_{even}[n]$ is a period 4 synthesized waveform, as is $x_{odd}[n]$

$\Rightarrow$ create a period 8 signal by doing 8 adds & 8 multiples (1 for each n) on the 2 period 4 signals. to get $x[n]\ \forall n$.



	Total	
8 adds & multiplies	8	8
2 (4 adds & multiples)	8	8
4 (2 adds & multiples)	8	8
0 adds & multiplies		

In general, if there are 2^M points, I have M levels of the tree & 2^M operations at each level of the tree.

So total number of operations is $M2^M$.

If I denote $2^M = N$, I have an N-point FFT so the total number of operations is

$N\log_2 N$.

(4) Suppose I convolve $h[n]$ with $x[n]$, each of which has a finite number of samples N.

Convolution takes $\frac{N(N-1)}{2}$ add & multiplies, $\Rightarrow O(N^2)$.

If I do two FFTs, 1 multiply, & 1 inverse FFT, I take $3 \cdot 2^N \log_2 2^N + 2N$,
$= O(N) \ll O(N^2)$

eg) $N = 10^4$, $N^2 = 10^8$. $\log_2 10^4 = 4 \log_2 10 \approx 13$. $\Rightarrow 13 \times 10^4$ operations.

In general, matrix multiplication is $O(N^3)$, but here we are exploiting both repetition (structure) & ~~re~~ periodicity.