6.003R

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Example

Given $x[n]$, $X(e^{j\Omega})$

$$y[n] = x[-n]$$

Find $Y(e^{j\Omega})$

$$Y(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} e^{-j\Omega n}\, y[n]$$

$$= \sum_{n=-\infty}^{\infty} e^{-j\Omega n}\, x[-n]$$

$$m = -n$$

$$= \sum_{m=\infty}^{-\infty} e^{j\Omega m}\, x[m]$$

$$= \sum_{m=-\infty}^{\infty} e^{-j(-\Omega)m}\, x[m]$$

$$= X(e^{-j\Omega})$$

Example

$$\sum_{n=0}^{N-1} x^*[n]\, y[n] = \, ?$$

$$x[n] = \sum_{k=0}^{N-1} a_k\, e^{j\frac{2\pi}{N}kn}$$

$$y[n] = \sum_{k=0}^{N-1} b_k\, e^{j\frac{2\pi}{N}kn}$$

$$= \sum_{n=0}^{N-1} \left[\sum_{k=0}^{N-1} a_k\, e^{j\frac{2\pi}{N}kn}\right]^* \left[\sum_{k'=0}^{N-1} b_{k'}\, e^{j\frac{2\pi}{N}k'n}\right]$$

$$= \sum_{k=0}^{N-1} \sum_{k'=0}^{N-1} a_k^* b_{k'} \underbrace{\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(k'-k)n}}_{\substack{= N \quad k = k' \\ = 0 \quad k \neq k'}}$$

$$= N \sum_{k=0}^{N-1} a_k^* b_k$$

Example

$$a_k = \begin{cases} 1 & k \text{ even} \\ 0 & k \text{ odd} \end{cases}$$

$$x[n] = \sum_k a_k e^{j\frac{2\pi}{N}kn}$$

First, let N be even. What might we expect? Distance between diff frequency components doubled, so probably half the frequency in discrete time.

$$a_k = \frac{1}{2}\left[1 + (-1)^k\right] = \frac{1}{2}\left[1 + e^{j\pi k}\right]$$

We expect the $\frac{1}{2}$ to be connected with $\delta[n]$

$$a_k = \frac{1}{N}\sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}kn} x[n] = \frac{1}{N}\sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}kn}\delta[n]$$

$$= \frac{1}{N}$$

$$\delta[n] \longleftrightarrow \frac{1}{N}$$

$$\frac{N}{2}\delta[n] \longleftrightarrow \frac{1}{2}$$

What about other term? try $\delta[n-\frac{N}{2}]$

$$a_k = \frac{1}{N}\sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}kn}\,\delta[n-\tfrac{N}{2}]$$

$$= \frac{1}{N} e^{-j\frac{2\pi}{N}k\frac{N}{2}} = \frac{1}{N} e^{-j\pi k}$$

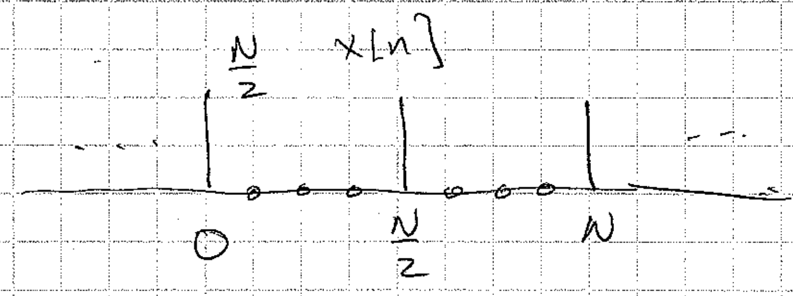
$$= \frac{1}{N}(-1)^k$$

$$\delta[n-\tfrac{N}{2}] \longleftrightarrow \frac{1}{N}(-1)^k$$

$$\frac{N}{2}\delta[n-\tfrac{N}{2}] \longleftrightarrow \frac{(-1)^k}{2}$$

So if $a_k = \frac{1}{2}[1+(-1)^k]$

$$x[n] = \frac{N}{2}\left[\delta[n] + \delta[n-\tfrac{N}{2}]\right]$$



What happens if N is odd?

Then the problem gets more interesting (translation: harder)

The big problem is that we cannot make $(-1)^k$ easily any more...

$$\delta[n] \longleftrightarrow \frac{1}{N}\sum_{k=0}^{N-1} e^{-j\frac{2\pi}{N}kn}\,\delta[n] = \frac{1}{N}$$

$$\delta\left[n-\tfrac{N+1}{2}\right] \longleftrightarrow \frac{1}{N}\sum_{k=0}^{N-1} e^{-j\frac{2\pi}{N}kn}\,\delta\left[n-\tfrac{N+1}{2}\right]$$

$$= \frac{1}{N}e^{-j\frac{2\pi}{N}k\left(\frac{N+1}{2}\right)}$$

$$= \frac{1}{N}e^{-j\pi k\left(1+\frac{1}{N}\right)}$$

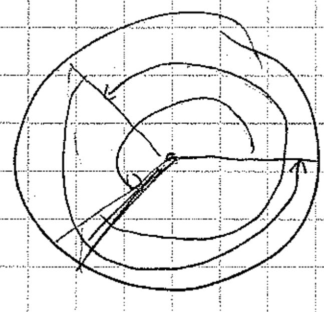
$$\delta\left[n-\tfrac{N-1}{2}\right] \Longleftrightarrow \frac{1}{N}e^{-j\pi k\left(1-\frac{1}{N}\right)}$$

So, what can we do? To get intuition, we can look at specific cases...

<u>$N = 3$</u>

$$x[n] = e^{j\frac{2\pi}{N}0n} + e^{j\frac{2\pi}{N}2n}$$

$$= 1 + e^{j\frac{4\pi}{3}n} = 1 + \cos\left(\tfrac{4\pi}{3}n\right) + j\sin\left(\tfrac{4\pi}{3}n\right)$$



n	$\cos\left(\frac{4\pi}{3}n\right)$	$\sin\left(\frac{4\pi}{3}n\right)$	$x[n]$
0	1	0	2
1	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2} - j\frac{\sqrt{3}}{2}$
2	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2} + j\frac{\sqrt{3}}{2}$

Try $N = 5$

$$x[n] = 1 + e^{j\frac{4\pi}{5}n} + e^{j\frac{8\pi}{5}n}$$

n	$x[n]$
0	3
1	$\frac{1}{2} - 0.363j$
2	$\frac{1}{2} - 1.5388j$
3	$\frac{1}{2} + 1.5388j$
4	$\frac{1}{2} + 0.363j$

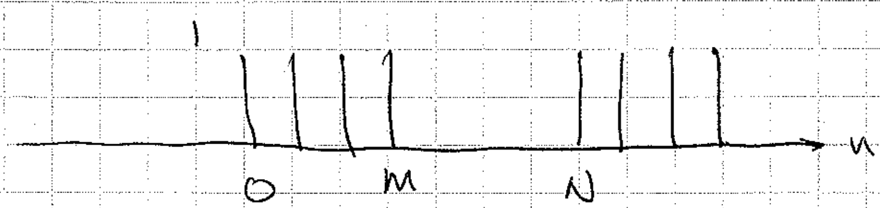
Try $N=7$

$$x[n] = 1 + e^{j\frac{4\pi}{7}n} + e^{j\frac{8\pi}{7}n} + e^{j\frac{12\pi}{7}n}$$

n	$x[n]$
0	4
1	$\frac{1}{2} - 0.241j$
2	$\frac{1}{2} - 0.627j$
3	$\frac{1}{2} - 2.191j$
4	$\frac{1}{2} + 2.191j$
5	$\frac{1}{2} + 0.627j$
6	$\frac{1}{2} + 0.241j$

Interesting, perhaps unexpected...

Example



Periodic discrete signal x[n]

$$N a_k = \sum_{k=0}^{N-1} e^{-j\frac{2\pi}{N}kn} \, x[n]$$

$$= 1 + e^{-j\frac{2\pi}{N}k} + e^{-j\frac{2\pi}{N}(2k)} + \cdots e^{-j\frac{2\pi}{N}Mk}$$

$$\longrightarrow \frac{1 - e^{-j\frac{2\pi}{N}(M+1)k}}{1 - e^{-j\frac{2\pi}{N}k}}$$

Want it as shifted sinc-type function appropriate for DT

$$= \frac{e^{-j\frac{2\pi}{N}\frac{(M+1)}{2}k}}{e^{-j\frac{2\pi}{N}\frac{k}{2}}} \; \frac{e^{j\frac{2\pi}{N}\frac{(M+1)}{2}k} - e^{-j\frac{2\pi}{N}\left(\frac{M+1}{2}\right)k}}{e^{j\frac{2\pi}{N}\frac{k}{2}} - e^{-j\frac{2\pi}{N}\frac{k}{2}}}$$

$$= e^{-j\frac{2\pi}{N}\frac{M}{2}k} \; \frac{\sin\left[\frac{\pi(M+1)k}{N}\right]}{\sin\left[\frac{\pi k}{N}\right]}$$

shift to Right by $\frac{M}{2}$

~~cent~~ sinc-type function for centered set of Delta's

OK, suppose now that

$$y[n] = \sum_{k=0}^{N-1} b_k \, e^{-j\frac{2\pi}{N}kn}$$

$$\text{with } b_k = a_k^2 = \left| e^{-j\frac{2\pi}{N}\frac{M}{2}k} \frac{\sin\left[\frac{\pi(M+1)k}{N}\right]}{\sin\left[\frac{\pi}{N}k\right]} \right|^2$$

Then expect $y[n] \sim \cancel{x[n] * x[n]}\ (x * x)[n]$

OK, check to be sure

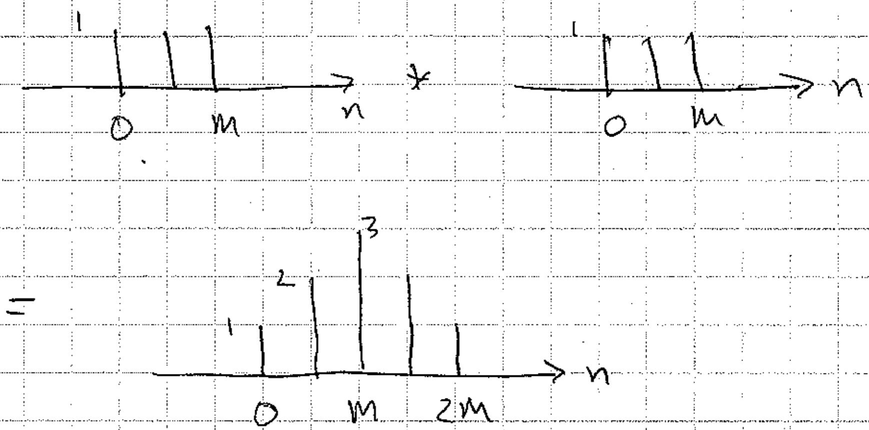
$$y[n] = \sum_{k=0}^{N-1} a_k^2 \, e^{j\frac{2\pi}{N}kn}$$

$$= \sum_{k=0}^{N-1} \left[\frac{1}{N}\sum_{n'=0}^{N-1} e^{-j\frac{2\pi}{N}kn'} x[n']\right]\left[\frac{1}{N}\sum_{n''=0}^{N-1} e^{-j\frac{2\pi}{N}kn''} x[n'']\right] e^{j\frac{2\pi}{N}kn}$$

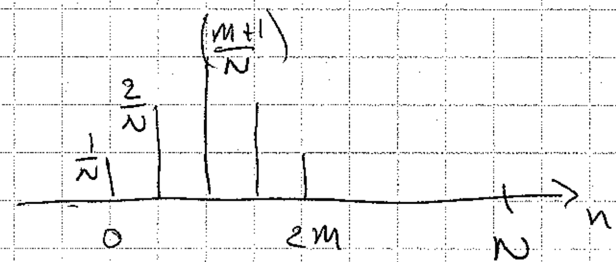
$$= \frac{1}{N^2}\sum_{n'}^{N-1}\sum_{n''}^{N-1} x[n']x[n''] \underbrace{\sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}k(n-n'-n'')}}_{\substack{N \quad n = n'+n'' \\ 0 \quad n \neq n'+n''}}$$

$$= \frac{1}{N}\sum_{n'}^{N-1} x[n']\,x[n-n'] = \frac{1}{N}(x * x)[n]$$

If $2M < N$, then we get simple convolution



So $y[n]$ =



If $2M > N$, then we get contributions from outside