6.003R April 21, 2010

Overview

Laplace transform

$$x(t) = \int_{\sigma-j\infty}^{\sigma+j\infty} X(s) e^{st} \frac{ds}{2\pi j} \qquad X(s) = \int_{-\infty}^{\infty} e^{-st} x(t)\, dt$$

Fourier transform

$$x(t) = \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} \frac{d\omega}{2\pi} \qquad X(j\omega) = \int_{-\infty}^{\infty} e^{-j\omega t} x(t)\, dt$$

Fourier series for periodic signals

$$x(t) = \sum_k a_k e^{j\frac{2\pi}{T}kt} \qquad a_k = \frac{1}{T}\int_T e^{-j\frac{2\pi}{T}kt} x(t)\, dt$$

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Z- transform

$$x[n] = \oint X(z)\, z^{n-1} \frac{dz}{2\pi j} \qquad X(z) = \sum_{n=-\infty}^{\infty} z^{-n}\, x[n]$$

Discrete time Fourier transform

$$x[n] = \int_{2\pi} X(e^{j\Omega})\, e^{j\Omega n} \frac{d\Omega}{2\pi} \qquad X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} e^{-j\Omega n}\, x[n]$$

Periodic discrete time series

$$x[n] = \sum_{k=0}^{N-1} a_k\, e^{j\frac{2\pi}{N}kn} \qquad a_k = \frac{1}{N}\sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}kn}\, x[n]$$

Numerical differentiation using FFT

Start with expansion of $x(t)$

$$x(t) = \int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t} \frac{d\omega}{2\pi}$$

Note that taking derivatives in time is equivalent to multiplication by $(j\omega)$ factors in frequency

$$\frac{dx}{dt} = \int_{-\infty}^{\infty} \left[j\omega X(j\omega) \right] e^{j\omega t} \frac{d\omega}{2\pi}$$

$$\frac{d^2x}{dt^2} = \int_{-\infty}^{\infty} \left[(j\omega)^2 X(j\omega) \right] e^{j\omega t} \frac{d\omega}{2\pi}$$

Approach:

Use mapping between t and n

$$t = nT \longleftrightarrow n$$

$$\frac{dx}{dt} = \frac{dn}{dt}\frac{dx}{dn} = \frac{1}{T}\frac{dx}{dn}$$

$$\frac{d^2x}{dt^2} = \frac{1}{T^2}\frac{d^2x}{dn^2}$$

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Algorithm

(1) Discretize using $\tilde{x}[n] = x(nT)$

(2) Compute

$$a_k = \frac{1}{N} \sum_{n=0}^{N-1} e^{-j\left(\frac{2\pi}{N}\right)kn} \tilde{x}[n] \quad \text{using FFT}$$

(3) Compute

$$\frac{d\tilde{x}}{dn} = \sum_{k=0}^{N-1} \left(j \frac{2\pi}{N} k\right) a_k \, e^{j\left(\frac{2\pi}{N}\right)kn} \quad \text{using FFT}$$

$$\frac{d^2\tilde{x}}{dn^2} = \sum_{k=0}^{N-1} \left(j \frac{2\pi}{N} k\right)^2 a_k \, e^{j\left(\frac{2\pi}{N}\right)kn} \quad \text{using FFT}$$

(4) Reconstruct continuous time points

$$\left(\frac{dx}{dt}\right)_{t=nT} \approx \frac{1}{T}\left(\frac{d\tilde{x}}{dn}\right)_n$$

$$\left(\frac{d^2x}{dt^2}\right)_{t=nT} \approx \frac{1}{T^2}\left(\frac{d^2\tilde{x}}{dn^2}\right)_n$$

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Thinking about numerical error

The FFT allows us to evaluate $\frac{d^2\tilde{x}}{du}$ and $\frac{d^2\tilde{x}}{du^2}$ seemingly without error (but there will be round-off error).

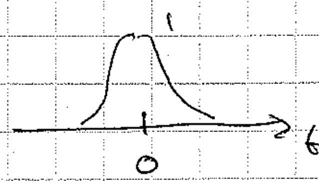
The only question then is how good of an approximation is ~~$\tilde{x}[n]$ to $x[n]$ $x(t)$~~ $\tilde{x}[n]$ (as a continuous function of u) in the vicinity of $x(nT)$ to $x(t)$.

Would expect it to have an accuracy comparable to what we would get from N-point Lagrange interpolation and then analytic integration (which becomes very good if lots of points are used)

Basically, if you undersample so that you don't have enough points to get a good approximation to the continuous function, the accuracy will be poor

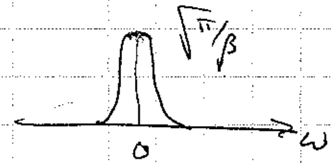
Example

let $x(t) = e^{-\beta t^2}$

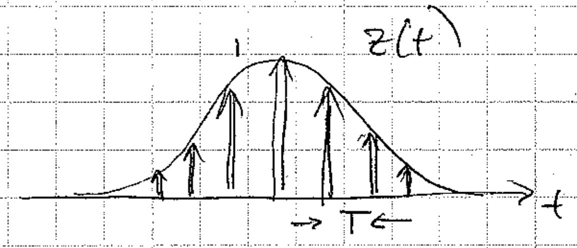


$$X(j\omega) = \int_{-\infty}^{\infty} e^{-j\omega t} x(t)\, dt$$

$$= \int_{-\infty}^{\infty} e^{-j\omega t} e^{-\beta t^2} dt = \sqrt{\frac{\pi}{\beta}}\, e^{-\omega^2/4\beta}$$



Consider



$$z(t) = e^{-\beta t^2} \sum_n \delta(t - nT)$$

Product in time implies convolution in frequency

$$x(t) = e^{-\beta t^2} \longleftrightarrow X(j\omega) = \sqrt{\frac{\pi}{\beta}}\, e^{-\omega^2/4\beta}$$

$$y(t) = \sum_n \delta(t - nT) \longleftrightarrow Y(j\omega) = \sum_k \frac{2\pi}{T} \delta\left(\omega - \frac{2\pi k}{T}\right)$$

$$z(t) = x(t)\, y(t)$$

$$Z(j\omega) = \frac{1}{2\pi} X(j\omega) * Y(j\omega)$$

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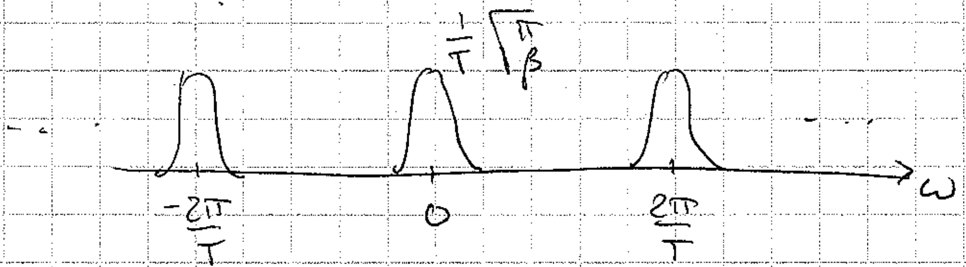
$$Z(j\omega) = \frac{1}{2\pi} \sum_k \sqrt{\frac{\pi}{\beta}} \, \frac{2\pi}{T} \, e^{-(\omega - \frac{2\pi}{T}k)^2/4\beta}$$

$$= \frac{1}{T} \sqrt{\frac{\pi}{\beta}} \sum_k e^{-(\omega - \frac{2\pi}{T}k)^2/4\beta}$$



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Thinking about filters

$$x(t) \longrightarrow \boxed{H(j\omega)} \longrightarrow y(t)$$

Four different cases

CT signal

$$y(t) = \int_{-\infty}^{\infty} H(j\omega)\, X(j\omega)\, e^{j\omega t} \frac{d\omega}{2\pi}$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} e^{-j\omega t'} h(t')\, dt'\right]\left[\int_{-\infty}^{\infty} e^{-j\omega t''} x(t'')\, dt''\right] e^{j\omega t} \frac{d\omega}{2\pi}$$

$$= \int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} dt''\; h(t')\, x(t'') \underbrace{\int_{-\infty}^{\infty} e^{j\omega(t-t'-t'')} \frac{d\omega}{2\pi}}_{\delta(t-t'-t'')}$$

$$= \int_{-\infty}^{\infty} h(t')\, x(t-t')\, dt$$

$$\boxed{y(t) = (h * x)(t)}$$

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Periodic CT signal

$$x(t) = \sum_k a_k e^{j\frac{2\pi}{T}kt}$$

$$e^{j\frac{2\pi}{T}kt} \longrightarrow \boxed{H(j\omega)} \longrightarrow H\left(j\tfrac{2\pi}{T}k\right) e^{j\frac{2\pi}{T}kt}$$

So

$$y(t) = \sum_k H\left(j\tfrac{2\pi}{T}k\right) a_k e^{j\frac{2\pi}{T}kt}$$

Would like to see this lead to a convolution

$$= \sum_k \int_{-\infty}^{\infty} e^{-j\frac{2\pi}{T}kt'} h(t')\,dt' \;\frac{1}{T}\int_{T} e^{-j\frac{2\pi}{T}kt''} x(t'')\,dt'' \; e^{j\frac{2\pi}{T}kt}$$

$$= \int_{-\infty}^{\infty} dt' \;\frac{1}{T}\int_{T} dt'\; h(t')x(t'') \sum_k e^{j\frac{2\pi}{T}k(t-t'-t'')}$$

Thinking about ~~sumat~~ summation

$$\sum_k \frac{1}{T} e^{j\frac{2\pi}{T}k(t-t'-t'')} = \sum_n \delta(t-t'-t''-nT)$$

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$$y(t) = \int_{-\infty}^{\infty} dt' \int_T dt''\, h(t')\, x(t'') \sum_n \delta(t-t'-t''-nT)$$

Can understand result most easily if we integrate over t'

$$y(t) = \sum_n \int_T x(t'')\, h(t-t''-nT)\, dt''$$

This is interesting...

$$= \int_T x(t')\, h(t-t')\, dt'$$

$$+ \int_T x(t')\, h(t-t'-T)\, dt'$$

$+ \ldots$ other responses

$$= \int_{-\infty}^{\infty} x(t')\, h(t-t')\, dt'$$

$$= (x * h)(t)$$

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Discrete time case

$$x[n] = \int_{2\pi} X(e^{j\Omega}) e^{j\Omega n} \frac{d\Omega}{2\pi}$$

$$e^{j\Omega n} \longrightarrow \boxed{H(\cancel{z})} \longrightarrow H(e^{j\Omega}) e^{j\Omega n}$$

$$y[n] = \int_{2\pi} H(e^{j\Omega}) X(e^{j\Omega}) e^{j\Omega n} \frac{d\Omega}{2\pi}$$

$$= \int_{2\pi} \left[\sum_{n'} e^{-j\Omega n'} h[n']\right]\left[\sum_{n''} e^{-j\Omega n''} x[n'']\right] e^{j\Omega n} \frac{d\Omega}{2\pi}$$

$$= \sum_{n'} \sum_{n''} h[n'] x[n''] \underbrace{\int_{2\pi} e^{j\Omega(n-n'-n'')} \frac{d\Omega}{2\pi}}_{\delta[n-n'-n'']}$$

$$= \sum_{n'} h[n'] x[n-n']$$

$$= (h * x)[n]$$

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Periodic discrete time case

$$x[n] = \sum_{k=0}^{N-1} a_k e^{i\frac{2\pi}{N}kn}$$

$$e^{i\frac{2\pi}{N}kn} \longrightarrow \boxed{H(z)} \longrightarrow H\left(e^{i\frac{2\pi}{N}k}\right) e^{i\frac{2\pi}{N}kn}$$

$$y[n] = \sum_{k=0}^{N-1} H\left(e^{i\frac{2\pi}{N}k}\right) a_k e^{i\frac{2\pi}{N}kn}$$

Once again, we would like to get convolution rule

$$= \sum_{k=0}^{N-1} \left[\sum_{n'=-\infty}^{\infty} e^{-i\frac{2\pi}{N}kn'} h[n'] \right] \left[\frac{1}{N} \sum_{n''=0}^{N-1} e^{-i\frac{2\pi}{N}kn''} x[n''] \right] e^{i\frac{2\pi}{N}kn}$$

$$= \sum_{n'=-\infty}^{\infty} \sum_{n''=0}^{N-1} h[n'] x[n''] \underbrace{\frac{1}{N} \sum_{k=0}^{N-1} e^{i\frac{2\pi}{N}k(n-n'-n'')}}_{\sum_m \delta[n-n'-n''-mN]}$$

$$= \sum_{m=-\infty}^{\infty} \sum_{n''=0}^{N-1} x[n''] h[n-n''-mN]$$

interesting... $= \sum_{n''=-\infty}^{\infty} x[n''] h[n-n''] = (x * h)[n]$

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