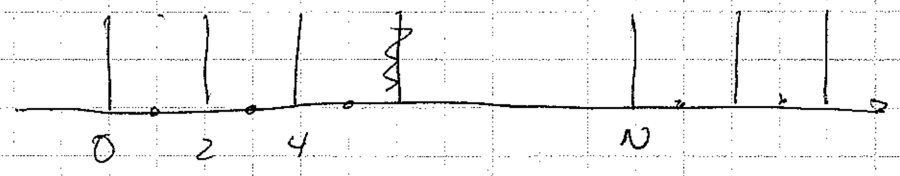
6.003 April 23, 2010

Example Pat Hagelt

$x[n]$, periodic



$$x[n] = \sum_{k=0}^{N-1} a_k \, e^{j\frac{2\pi}{N}kn}$$

$$a_k = \frac{1}{N}\sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}kn} x[n]$$

Want a_k:

First approach - just calculate it

$$a_k = \frac{1}{N}\left[1 + e^{-j\frac{2\pi}{N}(2k)} + e^{-j\frac{2\pi}{N}(4k)}\right]$$

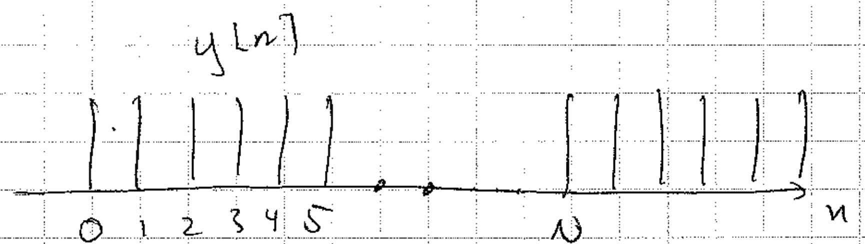
$$= \frac{1}{N}\;\frac{1 - e^{-j\frac{2\pi}{N}(6k)}}{1 - e^{-j\frac{2\pi}{N}(2k)}}$$

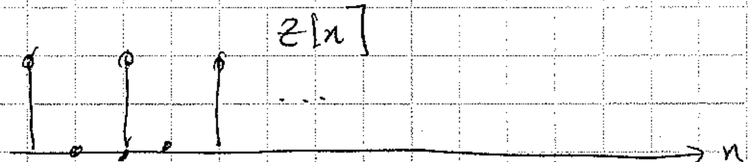
$$= \frac{1}{N}\;\frac{e^{-j\frac{2\pi}{N}(3k)}}{e^{-j\frac{2\pi}{N}k}}\;\frac{e^{j\frac{2\pi}{N}(3k)} - e^{-j\frac{2\pi}{N}(3k)}}{e^{j\frac{2\pi}{N}(k)} - e^{-j\frac{2\pi}{N}(k)}}$$

$$= \frac{1}{N}\,e^{-j\frac{2\pi}{N}(2k)}\;\frac{\sin\left[\frac{2\pi}{N}(3k)\right]}{\sin\left[\frac{2\pi}{N}(k)\right]}$$

Now, look at it as ~~convolution~~ product in time

$$x[n] = y[n]\, z[n]$$





$$y[n] = \sum_{k=0}^{N-1} b_k\, e^{j\frac{2\pi}{N}kn}$$

$$b_k = \frac{1}{N}\sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}kn}\, y[n]$$

$$= \frac{1}{N}\left[1 + e^{-j\frac{2\pi}{N}k} + \cdots\ e^{-j\frac{2\pi}{N}(5k)}\right]$$

$$= \frac{1}{N}\,\frac{1 - e^{-j\frac{2\pi}{N}(6k)}}{1 - e^{-j\frac{2\pi}{N}k}} = \frac{1}{N}\,\frac{e^{-j\frac{2\pi}{N}(3k)}}{e^{-j\frac{2\pi}{N}\left(\frac{k}{2}\right)}}\,\frac{e^{j\frac{2\pi}{N}(3k)} - e^{-j\frac{2\pi}{N}(3k)}}{e^{j\frac{2\pi}{N}\left(\frac{k}{2}\right)} - e^{-j\frac{2\pi}{N}\left(\frac{k}{2}\right)}}$$

$$= \frac{1}{N}\, e^{-j\frac{2\pi}{N}\left(\frac{5k}{2}\right)}\,\frac{\sin\left[\frac{2\pi}{N}(3k)\right]}{\sin\left[\frac{2\pi}{N}\left(\frac{k}{2}\right)\right]}$$

$$z[n] = \sum_{k=0}^{1} \cancel{e^{-j\frac{2\pi}{2}kn}} \, c_k \, e^{j\frac{2\pi}{2}kn}$$

$$c_k = \frac{1}{2} \sum_{n=0}^{1} e^{-j\frac{2\pi}{2}kn} \, z[n]$$

Period is 2

should not lead to problems

$$c_k = \frac{1}{2}[1] \quad \text{independent of } k$$

$$c_0 = \frac{1}{2} \quad c_1 = \frac{1}{2}$$

Convolution

$$x[n] = \sum_{k=0}^{N-1} a_k \, e^{-j\frac{2\pi}{N}kn}$$

$$a_k = \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}kn} \, x[n]$$

$$= \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}kn} \, y[n] \, z[n]$$

$$= \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}kn} \left[\sum_{k'=0}^{N-1} b_{k'} \, e^{j\frac{2\pi}{N}k'n} \right] \left[\sum_{k''=0}^{1} c_{k''} \, e^{j\frac{2\pi}{2}k''n} \right]$$

$$= \sum_{k'=0}^{N-1} \sum_{k''=0}^{1} b_{k'} c_{k''} \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(k'-k)n} \, e^{j\frac{2\pi}{2}k''n}$$

(3)

look at $k''=0$

$$\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(k'-k)} = N\,\delta[k'-k]$$

$k''=1$

$$\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(k'-k)n}\, e^{j\frac{2\pi}{2}k''n}$$

$$= \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(k'-k+\frac{N}{2}k'')}$$

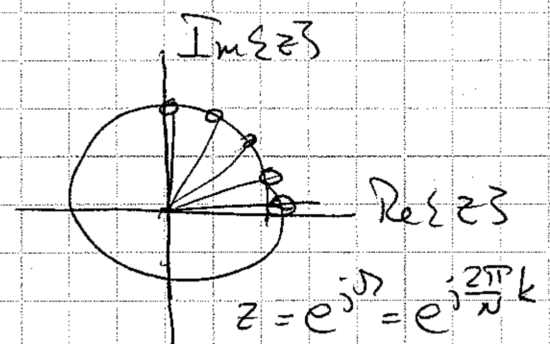
$$= N\,\delta[k'-k+\tfrac{N}{2}k'']$$

$$a_k = \sum_{k'=0}^{N-1} b_{k'}\,\frac{1}{2}\,\delta[k'-k]N$$

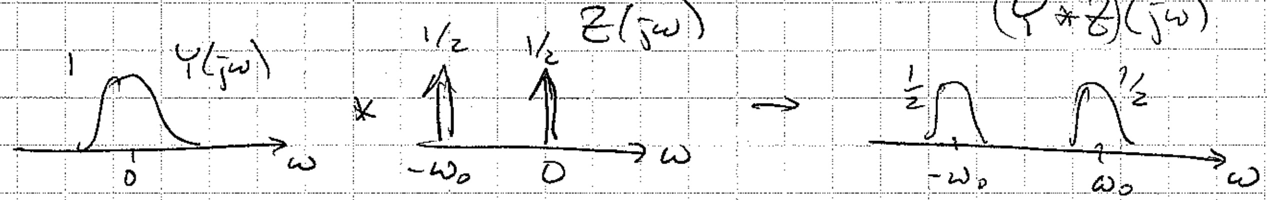
$$+ \sum_{k'=0}^{N-1} b_{k'}\,\frac{1}{2}\,\delta[k'-k+\tfrac{N}{2}]$$

$$= \frac{1}{2}b_k + \frac{1}{2}b_{k-\frac{N}{2}}$$

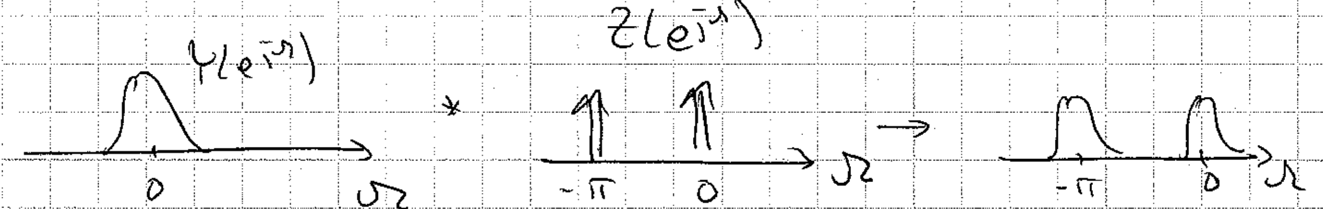
shift in k is shift in frequency from $\Omega=0$ to $\Omega=\pi$



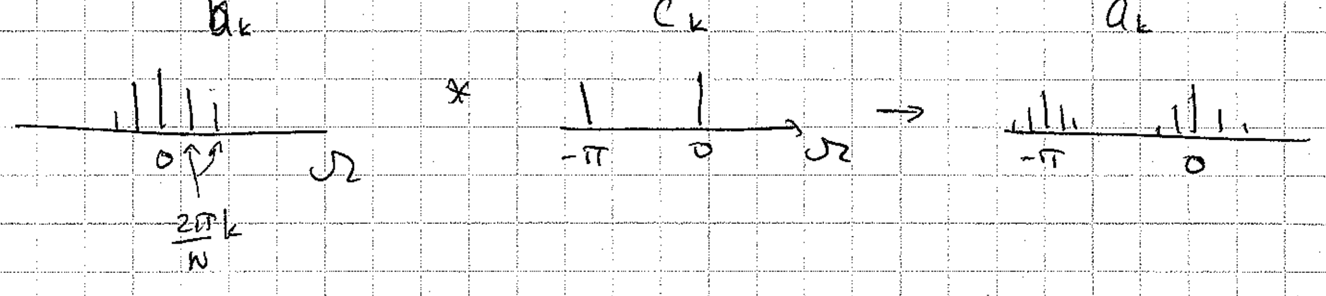
If this were CT problem, not periodic, then we would think



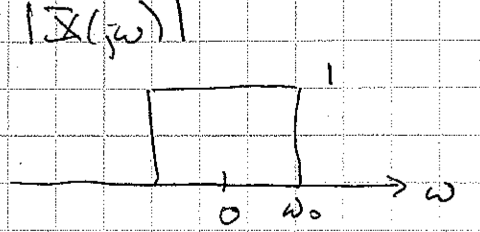
If this were DT problem, not periodic, then we would think

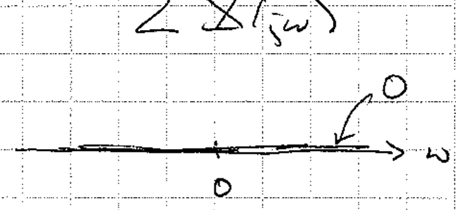


This DT problem that we considered is periodic, so that we can think of it as



Examples with change in phase





Find $x(t)$

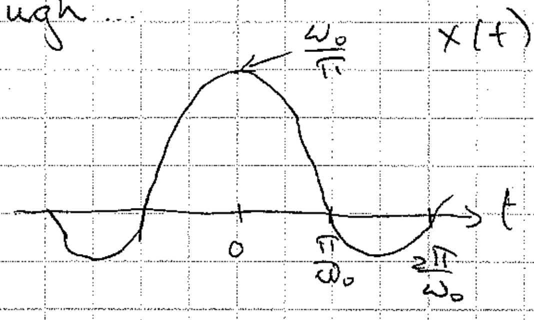
$$x(t) = \int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, \frac{d\omega}{2\pi}$$

$$= \int_{-\omega_0}^{\omega_0} e^{j\omega t}\, \frac{d\omega}{2\pi}$$

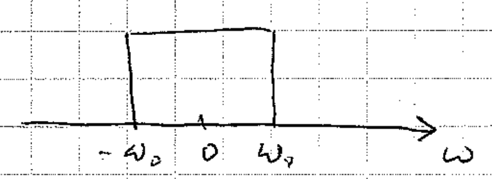
$$= \frac{1}{2\pi} \left. \frac{e^{j\omega t}}{jt} \right|_{-\omega_0}^{\omega_0} = \frac{1}{2\pi} \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{jt}$$

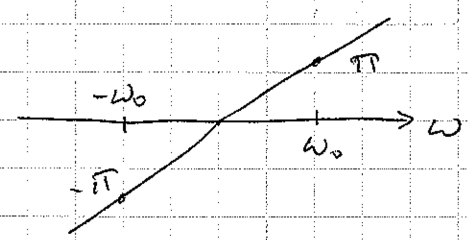
$$= \frac{\sin(\omega_0 t)}{\pi t}$$

OK, easy enough...



$|X(j\omega)|$ $\qquad\qquad$ $\angle X(j\omega)$





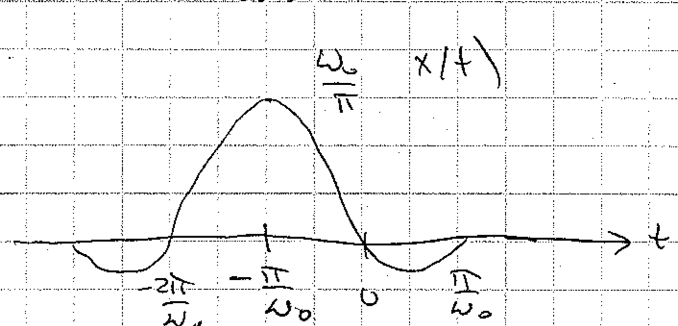
$$X(j\omega) = u(\omega+\omega_0)\,u(\omega_0-\omega)\,e^{j\pi\left(\frac{\omega}{\omega_0}\right)}$$

$$x(t) = \int_{-\infty}^{\infty} X(j\omega)\,e^{j\omega t}\,\frac{d\omega}{2\pi}$$

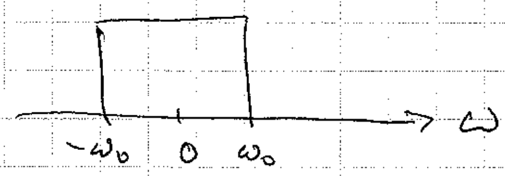
$$= \int_{-\omega_0}^{\omega_0} e^{j\pi\left(\frac{\omega}{\omega_0}\right)}\,e^{j\omega t}\,\frac{d\omega}{2\pi}$$

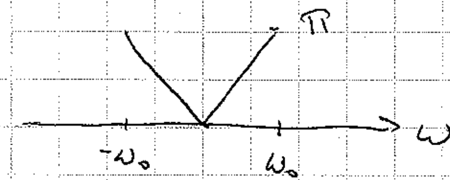
$$= \int_{-\omega_0}^{\omega_0} e^{j\omega\left(t+\frac{\pi}{\omega_0}\right)}\,\frac{d\omega}{2\pi}$$

$$= \frac{\sin \omega_0\left(t+\frac{\pi}{\omega_0}\right)}{\pi\left(t+\frac{\pi}{\omega_0}\right)}$$



$|X(j\omega)|$





$$X(j\omega) = u(\omega-\omega_0)u(-\omega)e^{-j\pi\frac{\omega}{\omega_0}}$$
$$+ u(\omega)u(\omega_0-\omega)e^{j\pi\frac{\omega}{\omega_0}}$$

$$x(t) = \int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\,\frac{d\omega}{2\pi}$$

$$= \int_0^{\omega_0} e^{+j\pi\frac{\omega}{\omega_0}}\, e^{j\omega t}\,\frac{d\omega}{2\pi}$$

$$+ \int_{-\omega_0}^{0} e^{-j\pi\frac{\omega}{\omega_0}}\, e^{j\omega t}\,\frac{d\omega}{2\pi}$$

$$= \int_0^{\omega_0} e^{j\omega\left(t+\frac{\pi}{\omega_0}\right)}\frac{d\omega}{2\pi} + \int_{-\omega_0}^{0} e^{j\omega\left(t-\frac{\pi}{\omega_0}\right)}\frac{d\omega}{2\pi}$$

$$= \frac{1}{2\pi}\left.\frac{e^{j\omega\left(t+\frac{\pi}{\omega_0}\right)}}{j\left(t+\frac{\pi}{\omega_0}\right)}\right|_0^{\omega_0} + \frac{1}{2\pi}\left.\frac{e^{j\omega\left(t-\frac{\pi}{\omega_0}\right)}}{j\left(t-\frac{\pi}{\omega_0}\right)}\right|_{-\omega_0}^{0}$$

$$\cancel{= \frac{1}{2\pi}\, e^{j\omega_0\left(t+\frac{\pi}{\omega_0}\right)} - 1}$$

$$= \frac{1}{2\pi}\,\frac{e^{j\omega_0\left(t+\frac{\pi}{\omega_0}\right)}-1}{j\left(t+\frac{\pi}{\omega_0}\right)} + \frac{1}{2\pi}\,\frac{1-e^{-j\omega_0\left(t-\frac{\pi}{\omega_0}\right)}}{j\left(t-\frac{\pi}{\omega_0}\right)}$$

$$= \frac{1}{2\pi} e^{j\frac{\omega_0}{2}\left(t+\frac{\pi}{\omega_0}\right)} \frac{\left[e^{j\frac{\omega_0}{2}\left(t+\frac{\pi}{\omega_0}\right)} - e^{-j\frac{\omega_0}{2}\left(t+\frac{\pi}{\omega_0}\right)} \right]}{j\left(t+\frac{\pi}{\omega_0}\right)}$$

$$+ \frac{1}{2\pi} e^{-j\frac{\omega_0}{2}\left(t-\frac{\pi}{\omega_0}\right)} \frac{\left[e^{j\frac{\omega_0}{2}\left(t-\frac{\pi}{\omega_0}\right)} - e^{-j\frac{\omega_0}{2}\left(t-\frac{\pi}{\omega_0}\right)} \right]}{j\left(t-\frac{\pi}{\omega_0}\right)}$$

$$= e^{j\frac{\omega_0}{2}\left(t+\frac{\pi}{\omega_0}\right)} \frac{\sin\left[\frac{\omega_0}{2}\left(t+\frac{\pi}{\omega_0}\right)\right]}{\pi\left(t+\frac{\pi}{\omega_0}\right)}$$

$$+ e^{-j\frac{\omega_0}{2}\left(t-\frac{\pi}{\omega_0}\right)} \frac{\sin\left[\frac{\omega_0}{2}\left(t-\frac{\pi}{\omega_0}\right)\right]}{\pi\left(t-\frac{\pi}{\omega_0}\right)}$$

$$\left[= j\, e^{j\frac{\omega_0 t}{2}} \frac{\sin\left[\frac{\omega_0}{2}\left(t+\frac{\pi}{\omega_0}\right)\right]}{\pi\left(t+\frac{\pi}{\omega_0}\right)} + j\, e^{-j\frac{\omega_0 t}{2}} \frac{\sin\left[\frac{\omega_0}{2}\left(t-\frac{\pi}{\omega_0}\right)\right]}{\pi\left(t-\frac{\pi}{\omega_0}\right)} \right]$$

What did phase modification do?

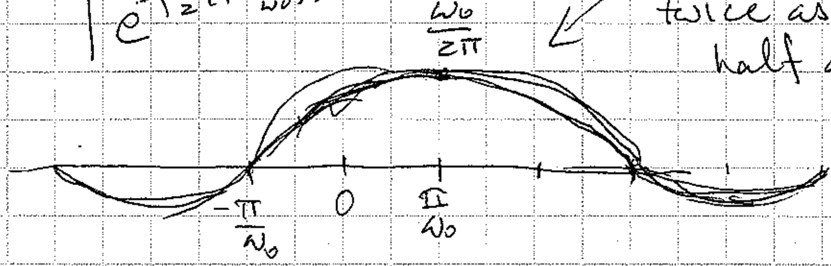
Split signal into 2 parts

linear phase corresponds to displacement in time

2 pieces are separated

$$\left| e^{-j\frac{\omega_0}{2}\left(t-\frac{\pi}{\omega_0}\right)} \sin \frac{\omega_0}{2}\left(t-\frac{\pi}{\omega_0}\right) / \pi\left(t-\pi/\omega_0\right) \right|$$

twice as wide, half as tall



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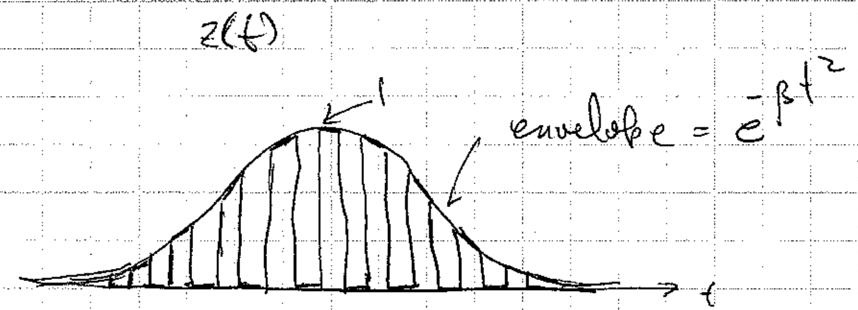
More complicated sampling



$$z(t) = x(t)\, y(t)$$

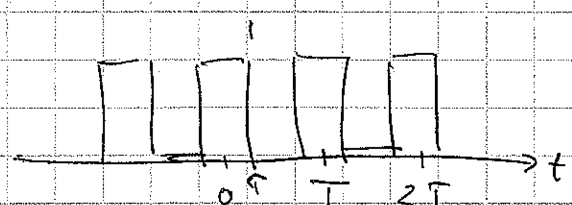
$$Z(j\omega) = \frac{1}{2\pi}(X * Y)(j\omega)$$

$$x(t) = e^{-\beta t^2} \qquad X(j\omega) = \int_{-\infty}^{\infty} e^{-j\omega t}\, x(t)\, dt$$

$$= \int_{-\infty}^{\infty} e^{-j\omega t}\, e^{-\beta t^2}\, dt$$

$$= \sqrt{\frac{\pi}{\beta}}\; e^{-\omega^2/4\beta}$$

y(t) =



Signal is periodic, so do Fourier series first,

Then get transform...

What is the Fourier series for this function?

$$y(t) = \sum_k b_k e^{j\frac{2\pi}{T}kt}$$

$$b_k = \frac{1}{T}\int_T e^{-j\frac{2\pi}{T}kt}\, y(t)\, dt$$

$$= \frac{1}{T}\int_{-\tau}^{\tau} e^{-j\frac{2\pi}{T}kt}\, dt$$

$$= \frac{1}{T}\left.\frac{e^{-j\frac{2\pi}{T}kt}}{\left(-j\frac{2\pi}{T}k\right)}\right|_{-\tau}^{\tau}$$

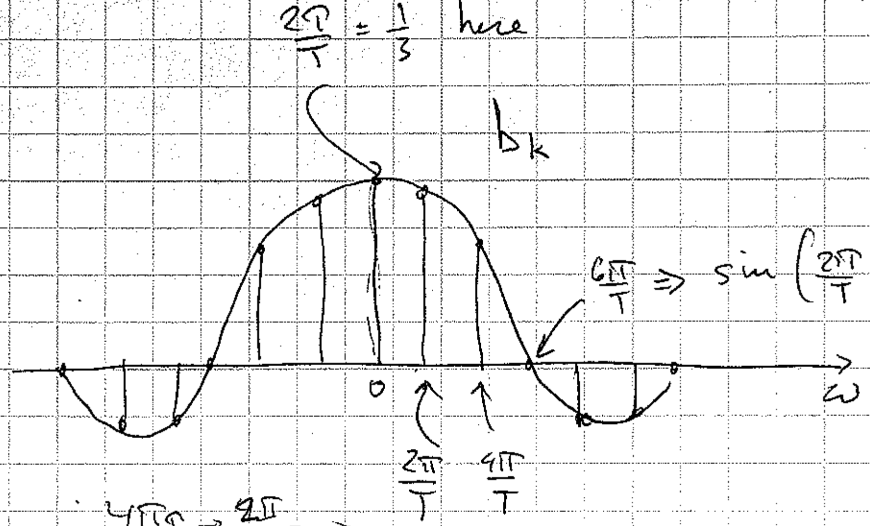
$$= \frac{1}{T}\,\frac{e^{-j\frac{2\pi}{T}k\tau} - e^{j\frac{2\pi}{T}k\tau}}{\left(-j\frac{2\pi}{T}k\right)}$$

$$= \frac{\sin\left(\frac{2\pi}{T}k\tau\right)}{\pi k}$$

OK, so what is $Y(j\omega)$?

$$Y(j\omega) = \sum_k 2\pi\, b_k\, \delta\left(\omega - \frac{2\pi}{T}k\right)$$

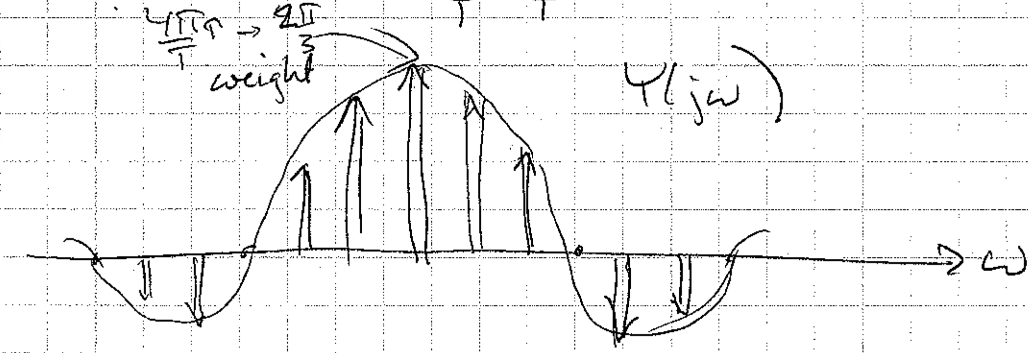
$$= \sum_k \frac{2}{k}\sin\left(\frac{2\pi}{T}k\tau\right)\delta\left(\omega - \frac{2\pi}{T}k\right)$$



$\frac{6\pi}{T} \Rightarrow \sin\left(\frac{2\pi}{T}k\tau\right) = \sin\pi$

so $\frac{\tau}{T}2 \cdot 3\pi = \pi$

$\frac{\tau}{T} = \frac{1}{6}$



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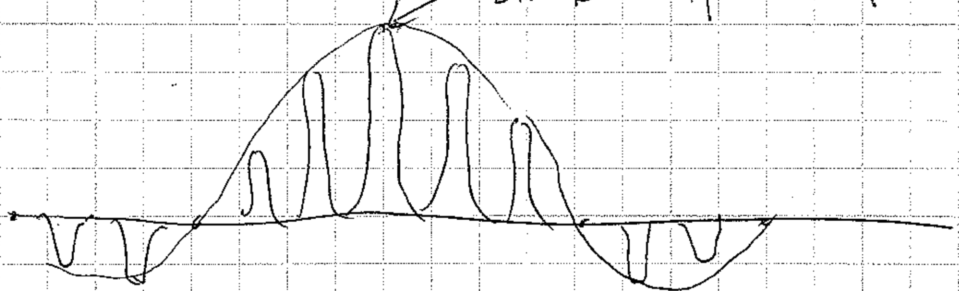
so, what does $Z(j\omega)$ looklike?

$$Z(j\omega) = \frac{1}{2\pi}(X * Y)(j\omega)$$

$\frac{1}{2\pi}\frac{2}{k}\frac{2\pi k\tau}{T}\sqrt{\frac{\pi}{\beta}} \to \frac{2\pi}{T}\sqrt{\frac{\pi}{\beta}} = \frac{1}{3}\sqrt{\frac{\pi}{\beta}}$



$$Z(j\omega) = \frac{1}{2\pi}\sum_k \frac{2}{k}\sin\left(\frac{2\pi}{T}k\tau\right)\sqrt{\frac{\pi}{\beta}}\, e^{-(\omega-\frac{2\pi}{T}k)^2/4\beta}$$