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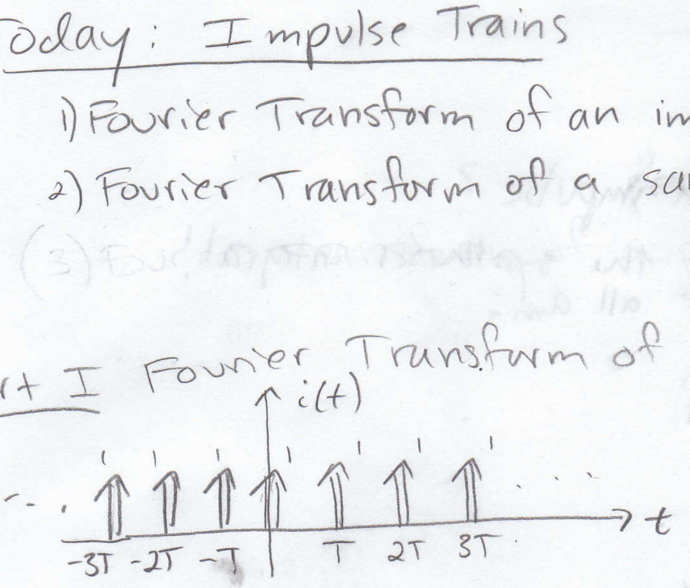
Recitation #20

4/23/10

Today: Impulse Trains

1) Fourier Transform of an impulse train
2) Fourier Transform of a sampled CT signal

Part I Fourier Transform of an impulse train



$$= \sum_{n=-\infty}^{+\infty} \delta(t-nT)$$

what is the Fourier transform? (CTFT)

By definition

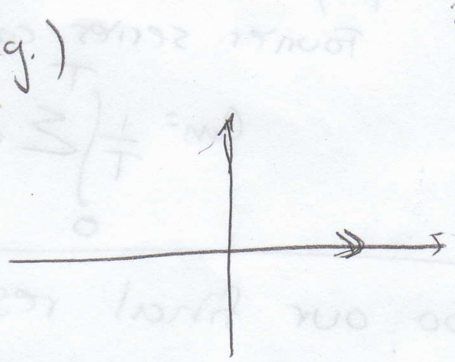
$$\int_{-\infty}^{\infty}\left(\sum_{n=-\infty}^{\infty}\delta(t-nT)\right)e^{-j2\pi f t}\,dt = \sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(t-nT)e^{-j2\pi f t}\,dt$$

$$= \sum_{n=-\infty}^{\infty} e^{-j2\pi f n T}$$

First, say $f=\frac{m}{T}$, m is an integer (pos. or neg.)

Then $I(f) = \sum_{n=-\infty}^{\infty} e^{-j2\pi n m}$

$$= \sum_{n=-\infty}^{\infty} 1 = 1+1+1+\cdots$$

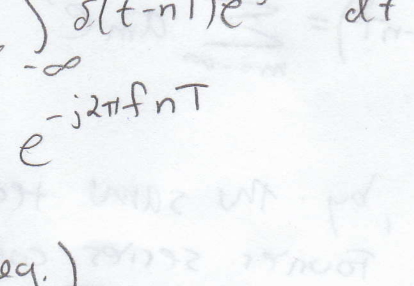


$$I\left(\frac{m}{T}\right) = \infty \,!$$

Now say $f=\frac{m+\epsilon}{T}$, $-1<\epsilon<1$

Then $I(f) = \sum_{n=-\infty}^{\infty} e^{-j2\pi\left(\frac{m+\epsilon}{\cancel{T}}\right)n\cancel{T}}$

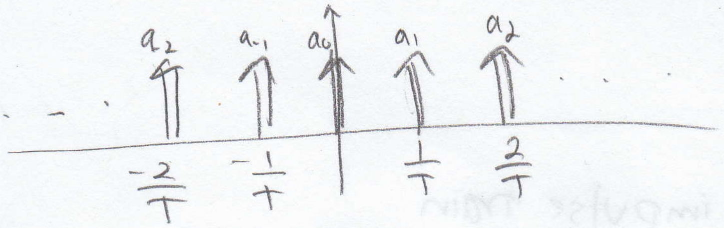
$$= \sum_{n=-\infty}^{\infty} e^{-j2\pi\epsilon n}$$



$I\left(\frac{m+\epsilon}{T}\right) = 0$ (pair every $e^{-j2\pi\epsilon n}$ vector with its conjugate vector, 180° version and cancel it out.)

* Note subtlety in rational (periodic cancellation) vs. irrational

Therefore $I(f)$ looks like



But what is the area of each impulse?
we can figure it out from the synthesis integral!
Assume they are all a_m:

$$= \int_{-\infty}^{\infty} \left(\sum_{m=-\infty}^{\infty} a_m \delta\left(f - \frac{m}{T}\right) \right) e^{j2\pi f t} \, df$$

$$= \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} a_m \delta\left(f - \frac{m}{T}\right) e^{j2\pi f t} \, df$$

$$\sum_{n=-\infty}^{\infty} \delta(t - nT) = \sum_{m=-\infty}^{\infty} a_m e^{j2\pi \frac{m}{T} t}$$

This looks like a Fourier series!

So, by the same technique that we used to get Fourier series coefficients

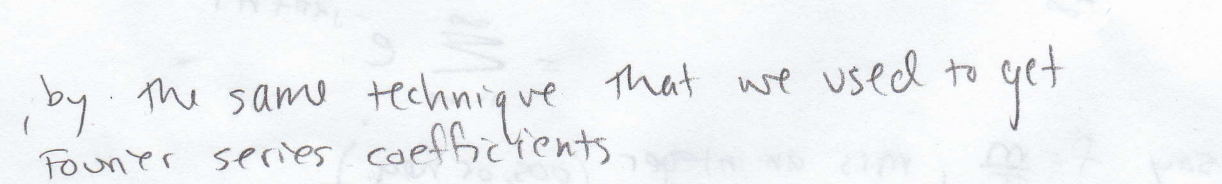
$$a_m = \frac{1}{T} \int_0^T \sum \delta(t - nT) \, dt = \frac{1}{T} \int_0^T \delta(t) \, dt = \frac{1}{T}, \; \forall m$$

So our final result is

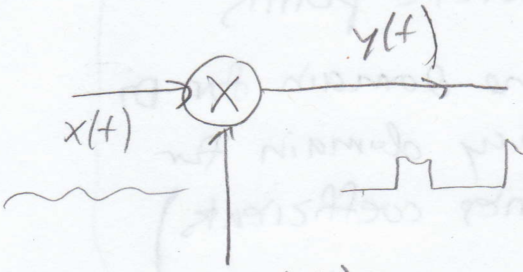
$$\mathcal{F}\left\{ \sum_{n=-\infty}^{\infty} \delta(t - nT) \right\} = \sum_{m=-\infty}^{\infty} \frac{1}{T} \delta\left(f - \frac{n}{T}\right)$$

impulse train

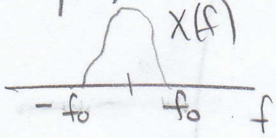
* note: if axis is ω $\frac{2\pi}{T}$

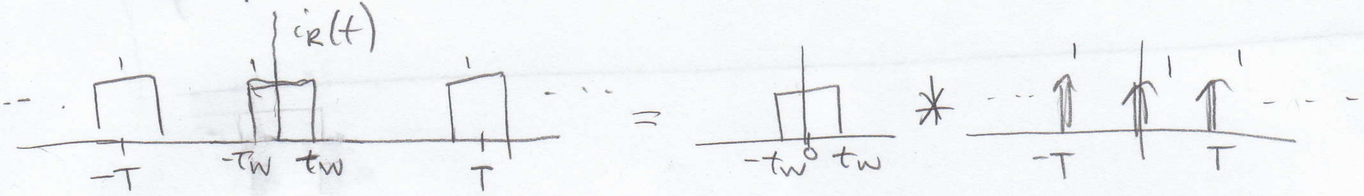


Part II: FT of a sampled CT signal

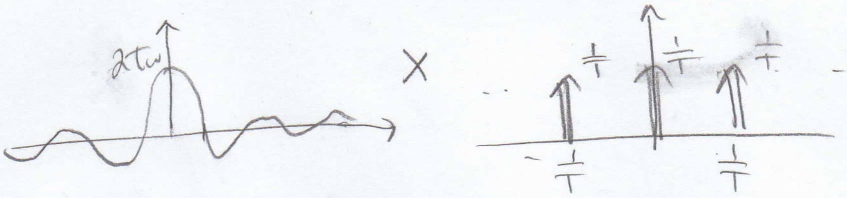


What is the spectrum of $y(t)$ if $X(f)$ is bandlimited





$I_R(f) =$



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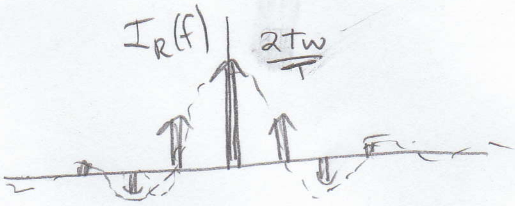
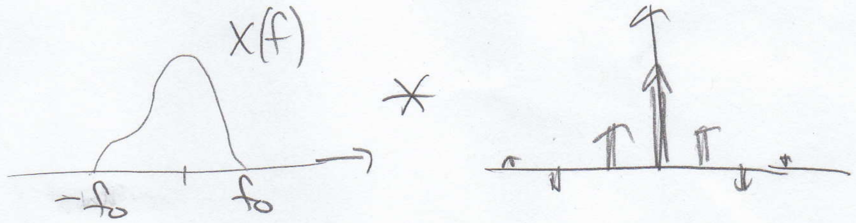


$Y(f) = X(f) * I_R(f)$



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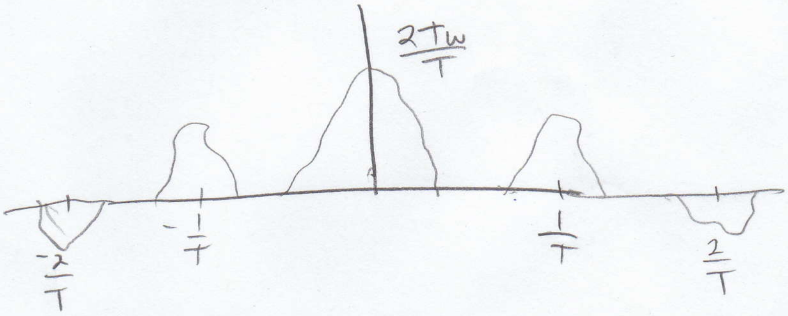
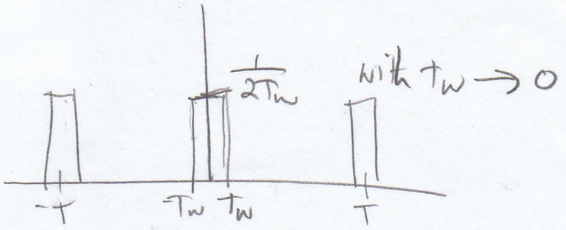
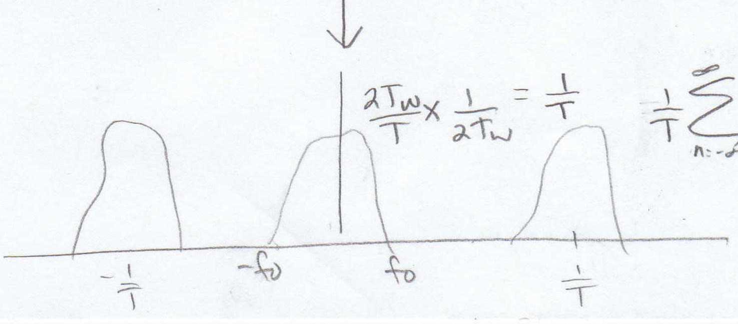


In the limit let



& then $i_R(t) = i(t)$ and we have an impulse train!



$\frac{1}{T}\sum_{n=-\infty}^{\infty} X\left(f - \frac{n}{T}\right)$

So Impulse areas = <u>values</u> at discrete points
(whether in the time domain for DT
or in the frequency domain for
CTFS (Fourier series coefficients))