6.003 R

April 30, 2010

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Overview

Digitization functionality:

$$x(t) \longrightarrow \boxed{\text{Sampling}} \longrightarrow x[n]$$

What to do with $x[n]$? Examples

$$x[n] \longrightarrow \boxed{\text{Dig. signal processing}} \longrightarrow y[n]$$

$$x[n] \longrightarrow \boxed{\text{Communication channel}} \longrightarrow y[n]\text{, at another location}$$

Going back to analog if needed

$$y[n] \longrightarrow \boxed{\text{Reconstruction}} \longrightarrow y(t)$$

Impulse sampling (Review)

Interested in different sampling relevant to HW problem. First like to review impulse sampling to understand better how to think about it.

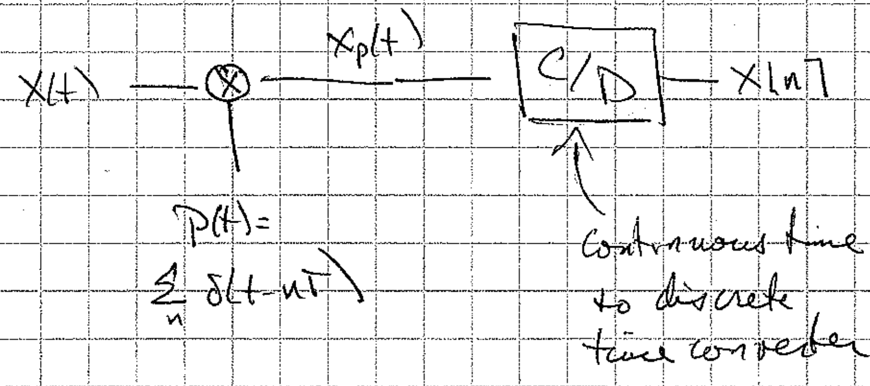
Sampling so far:

$$x[n] = x(nT)$$

DT signal CT signal

Note - other approaches possible, as in example we will do

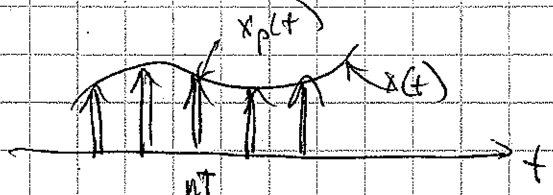
Approach outlined in text



Sampling with an impulse train

$$p(t) = \sum_n \delta(t-nT)$$

$$x_p(t) = x(t)\,p(t)$$



$$x_p(t) = \sum_n x(t)\,\delta(t-nT) = \sum_n x(nT)\,\delta(t-nT)$$

$$= \sum_n x[n]\,\delta(t-nT)$$

Converter:

$$x_p(t) \longrightarrow \boxed{C/D} \longrightarrow x[n]$$

$$\sum_n x[n]\,\delta(t-nT)$$

Converter takes the weights of the δ-functions to make up DT signal

Next, think about Fourier transforms

Start with Fourier transform of $x(t)$

$$X(j\omega) = \int_{-\infty}^{\infty} e^{-j\omega t} x(t)\,dt$$

This is what we start with. Like to relate transforms of $x_p(t)$ and $x[n]$ back to this one, to see what digitization does.

First, look at $X_p(j\omega)$

$$X_p(j\omega) = \int_{-\infty}^{\infty} e^{-j\omega t} x_p(t)\,dt$$

$$= \int_{-\infty}^{\infty} e^{-j\omega t} x(t)p(t)\,dt$$

$$= \frac{1}{2\pi}\left(X * P\right)(j\omega)$$

Do we remember $P(j\omega)$?

$$P(j\omega) = \sum_k \frac{2\pi}{T}\,\delta\left(\omega - \frac{2\pi}{T}k\right)$$

Check to be sure.. since $p(t)$ is periodic we can construct a series

4 $$p(t) = \sum_k a_k e^{i\frac{2\pi}{T}tk}$$

with $\quad a_k = \frac{1}{T}\int_T e^{-j\frac{2\pi}{T}kt}\, p(t)\, dt$

$$= \frac{1}{T}\int_{-T/2}^{T/2} e^{-j\frac{2\pi}{T}kt} \sum_k \delta(t-nT)\, dt$$

$$= \frac{1}{T}$$

So $\quad p(t) = \sum_k \frac{1}{T} e^{j\frac{2\pi}{T}kt} \;\left(= \sum_n \delta(t-nT)\right)$

We can check to see if Fourier transform is consistent

$$p(t) = \int_{-\infty}^{\infty} P(j\omega)\, e^{j\omega t}\, \frac{d\omega}{2\pi}$$

$$= \int_{-\infty}^{\infty} \sum_k \frac{2\pi}{T}\, \delta\!\left(\omega - \frac{2\pi}{T}k\right) e^{j\omega t}\, \frac{d\omega}{2\pi}$$

$$= \sum_k \frac{1}{T}\, e^{j\frac{2\pi}{T}kt} \Leftarrow$$ same as for series. so it is consistent

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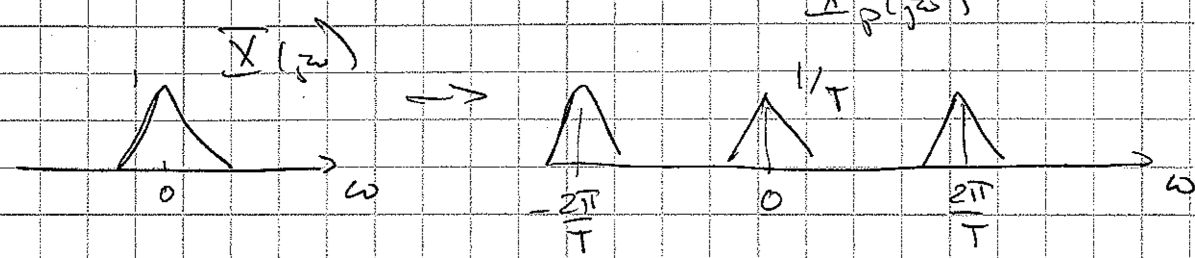
OK, so now we want convolution

$$X_p(j\omega) = \frac{1}{2\pi} P(j\omega) * X(j\omega)$$

$$= \frac{1}{2\pi}\left[\sum_k \frac{2\pi}{T}\delta\!\left(\omega - \frac{2\pi}{T}k\right)\right] * X(j\omega)$$

$$= \sum_k \frac{1}{T} X\!\left(j\left(\omega - \frac{2\pi}{T}k\right)\right)$$

Think about it in frequency



we get copies of $X(j\omega)$, scaled, and offset by integer multiples of $\frac{2\pi}{T}$.

Another approach to $X_p(j\omega)$:

We ~~know~~ know $X_p(j\omega)$ from the arguments above. There is another argument which is helpful, since it will get us $X(e^{j\Omega})$. Do brute force calculation

$$X_p(j\omega) = \int_{-\infty}^{\infty} e^{-j\omega t}\, x_p(t)\, dt$$

$$= \int_{-\infty}^{\infty} e^{-j\omega t}\left[\sum_n x[n]\,\delta(t-nT)\right] dt$$

$$= \sum_n x[n]\, e^{-j\omega nT}$$

But this is similar in form to

$$X(e^{j\Omega}) = \sum_n x[n]\, e^{-j\Omega n}$$

From this we know that

$$\underline{X}(e^{j\Omega}) = \underline{X}_p(j\omega)\Big|_{\omega T = \Omega} = \underline{X}_p\left(j\frac{\Omega}{T}\right)$$

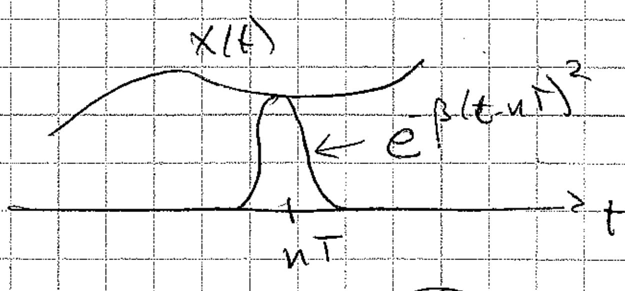
This is useful. Since $x_p(t)$ and $x[n]$ have Fourier transforms that are scaled versions of one another, we ~~can~~ can work with either one equivalently. We can use DT or CT approaches interchangeably.

Example: Sampling with a different function

Well, enough Review... We are ready for a problem something like the HW problem.

Consider Gaussian sampling defined by

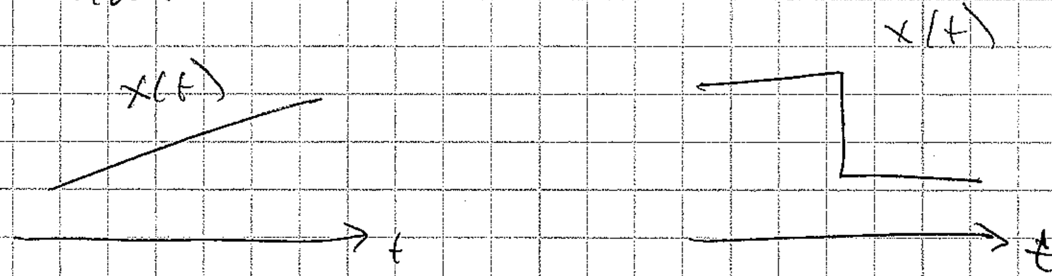
$$y[n] = \int_{-\infty}^{\infty} x(t)\, e^{\beta(t-nT)^2}\, dt$$



Before going on, what might we expect?

If ~~ga~~ Gaussian is narrow, then we would get a scaled version of $x[n] = x(nT)$. If Gaussian is broad, then we don't follow $x(t)$ so well.

Consider:



get $y[n] = x(nT) \times \text{const}$ if linear

$y[n]$ will be smoothed out version of scaled $x(t)$

So, we expect low pass filter functionality.

Can we show it?

We want to investigate transforms. We would like

$$Y(e^{j\Omega}) = \text{function of } [X(j\omega)]$$

So, roll up sleeves, and off we go...

$$Y(e^{j\Omega}) = \sum_n e^{-j\Omega n}\, y[n]$$

$$= \sum_n e^{-j\Omega n} \int_{-\infty}^{\infty} x(t)\, e^{-\beta(t-nT)^2}\, dt$$

Recall that $x(t) = \int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t} \frac{d\omega}{2\pi}$

$$= \sum_n e^{-j\Omega n} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, X(j\omega)\, e^{j\omega t}\, e^{-\beta(t-nT)^2}$$

So, we have $Y(e^{j\Omega})$ now explicitly in terms of $X(j\omega)$. So, can we simplify?

First integrate in time

$$\int_{-\infty}^{\infty} e^{j\omega t}\, e^{-\beta(t-nT)^2}\, dt = \underbrace{\sqrt{\frac{\pi}{\beta}}\, e^{-\omega^2/4\beta}}_{\text{Fourier transform of } e^{-\beta t^2}}\ \underbrace{e^{j\omega nT}}_{\text{shift in time factor}}$$

So

$$Y(e^{j\Omega}) = \sum_n e^{-j\Omega n} \int \frac{d\omega}{2\pi}\, X(j\omega) \sqrt{\frac{\pi}{\beta}}\, e^{-\omega^2/4\beta}\, e^{j\omega nT}$$

Next, sum in n

$$\sum_n e^{-j\Omega n} e^{j\omega n T} = 2\pi\,\delta(\Omega - \omega T)$$

$$+ 2\pi\,\delta(\Omega - \omega T - 2\pi)$$

$$+ \cdots$$

$$= \sum_k 2\pi\,\delta(\Omega - \omega T - 2\pi k)$$

$$Y(e^{j\Omega}) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, X(j\omega) \sqrt{\frac{\pi}{\beta}}\, e^{-\omega^2/4\beta} \sum_k 2\pi\,\delta(\Omega - \omega T - 2\pi k)$$

So, now we want to integrate in ω. But the δ-function has ωT in it. So best to scale frequency before integrating. Define

$$\eta = \omega T$$

$$d\eta = T\,d\omega$$

$$Y(e^{j\Omega}) = \int_{-\infty}^{\infty} X\left(j\frac{\eta}{T}\right) \sqrt{\frac{\pi}{\beta}}\, e^{-(\eta/T)^2/4\beta} \sum_k \delta(\Omega - \eta - 2\pi k)\, \frac{d\eta}{T}$$

$$= \frac{1}{T} \sum_k X\left(j\left(\frac{\Omega - 2\pi k}{T}\right)\right) \sqrt{\frac{\pi}{\beta}}\, e^{-\left(\frac{\Omega - 2\pi k}{T}\right)^2/4\beta}$$

Thinking about the answer...

If there is no aliasing, then we would expect the $k=0$ term to dominate

$$Y(e^{j\Omega}) \xrightarrow[\text{aliasing}]{\text{no}} \frac{1}{T} \underbrace{X\left(j\frac{\Omega}{T}\right)}_{\substack{\text{transform} \\ \text{of } x(t)}} \underbrace{\sqrt{\frac{\pi}{\beta}}\, e^{-(\Omega/T)^2/4\beta}}_{\substack{\text{transform of} \\ e^{-\beta t^2}}}$$

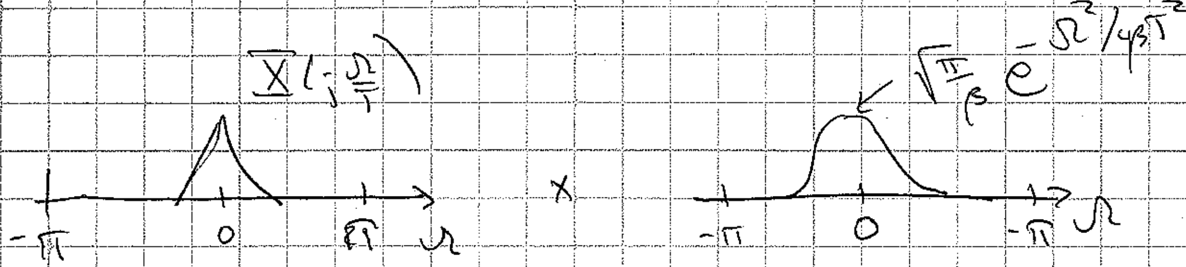
This funny product in frequency space suggest that we should look for a convolution in time. Go back to initial expression

$$y[n] = \int_{-\infty}^{\infty} x(t)\, e^{-\beta(t-nT)^2}\, dt$$

$$= \cancel{x(t) *} \left(x * e^{-\beta t^2}\right)(nT)$$

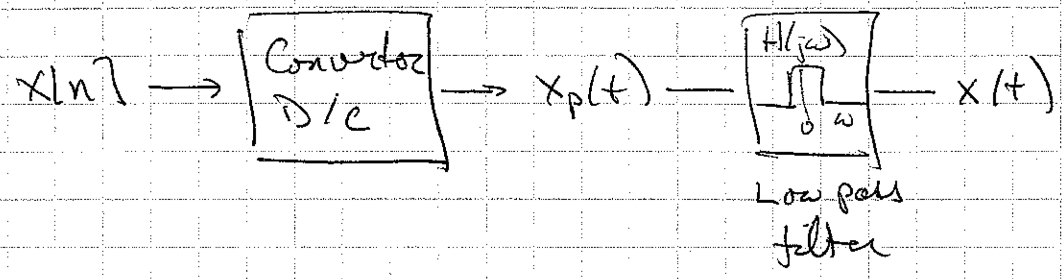
Sort of a funny convolution ~~ed~~ in time.

We can also see low pass filtering



again, in the limit that no aliasing occurs.

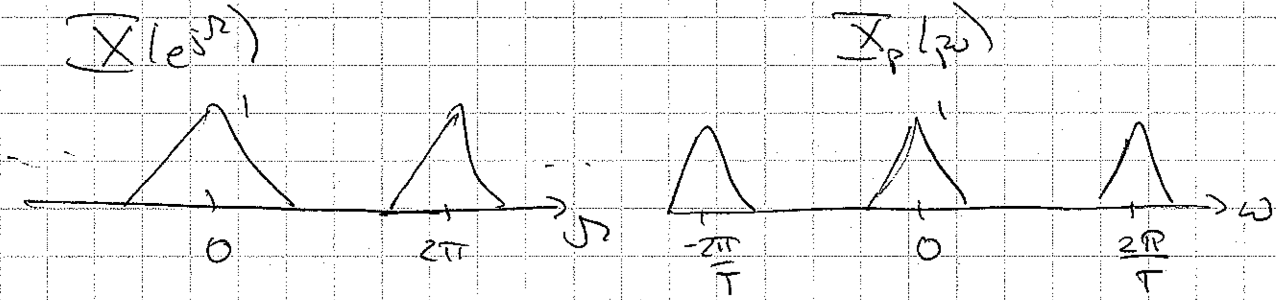
Example: Reconstruction, as discussed in class



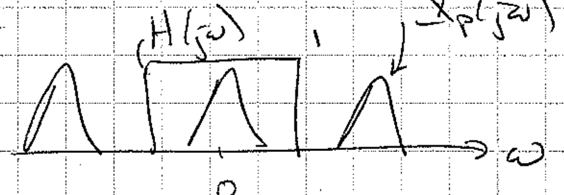
Look at transforms

$$X(e^{j\Omega}) = \sum_n e^{-j\Omega n} x[n]$$

$$x_p(t) = \sum_n x[n]\,\delta(t-nT)$$

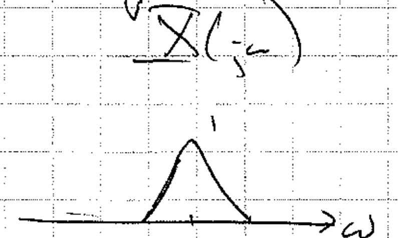


Now, ideal low pass filter



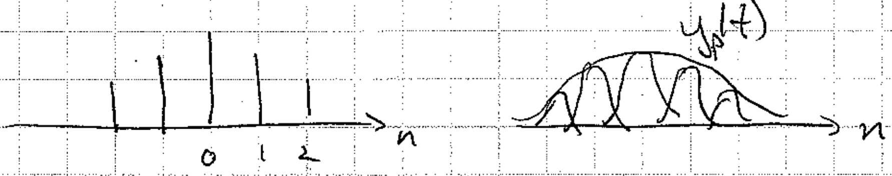
$\downarrow$

$X(j\omega)$



What about Reconstruction with Gaussians?

$$y_p(t) = \sum_n x[n]\, e^{-\beta(t-nT)^2}$$



$$Y(j\omega) = \int_{-\infty}^{\infty} e^{-j\omega t}\, y_p(t)\, dt$$

$$= \int_{-\infty}^{\infty} e^{-j\omega t} \left[\sum_n x[n]\, e^{-\beta(t-nT)^2} \right] dt$$

$$= \int_{-\infty}^{\infty} e^{-j\omega t} \sum_n \int_{2\pi} X(e^{j\Omega})\, e^{j\Omega n} \frac{d\Omega}{2\pi}\, e^{-\beta(t-nT)^2}\, dt$$

Integrate first in t

$$= \sum_n \int_{2\pi} \frac{d\Omega}{2\pi}\, X(e^{j\Omega})\, e^{j\Omega n} \underbrace{\int_{-\infty}^{\infty} dt\; e^{-j\omega t} e^{-\beta(t-nT)^2}}_{\sqrt{\frac{\pi}{\beta}}\, e^{-j\omega nT}\, e^{-\omega^2/4\beta}}$$

$$= \int_{2\pi} \frac{d\Omega}{2\pi}\, X(e^{j\Omega}) \sum_n e^{j\Omega n}\, e^{-j\omega nT} \sqrt{\frac{\pi}{\beta}}\, e^{-\omega^2/4\beta}$$

Next sum over n

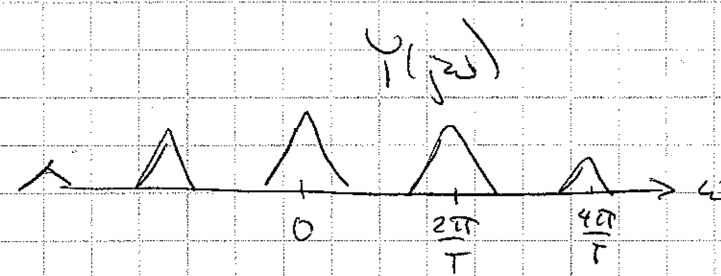
$$\sum_n e^{-j\Omega n} e^{-j\omega n T} = 2\pi\,\delta(\Omega - \omega T) + 2\pi\,\delta(\Omega - \omega T - 2\pi) + \cdots$$

$$= \sum_k 2\pi\,\delta(\Omega - \omega T - 2\pi k)$$

$$Y(j\omega) = \int_{2\pi} X(e^{j\Omega}) \sqrt{\frac{\pi}{\beta}}\, e^{-\omega^2/4\beta} \sum_k 2\pi\,\delta(\Omega - \omega T - 2\pi k) \frac{d\Omega}{2\pi}$$

$$= \sqrt{\frac{\pi}{\beta}}\, e^{-\omega^2/4\beta} \underbrace{\sum_k X\left(e^{j(\omega - \frac{2\pi}{T}k)T}\right)}_{\text{periodic train of } X(e^{j\Omega})}$$

envelope due to Gaussian



if Gaussian is spread out enough in time, then get filter effect in frequency.