6.003R May 7, 2010

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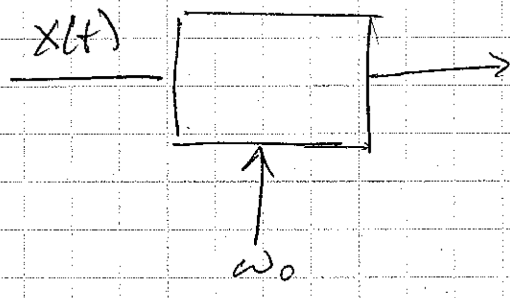
FM modulation



$y(t) = \cos \Theta(t)$

$$\frac{d\Theta}{dt} = \omega_0 + \alpha x(t)$$

$$\Theta(t) = \omega_0 t + \alpha \int^t x(t')\, dt'$$

Two limits:

(a) large modulation index $\Longrightarrow$ large bandwidth

$$\Theta(t) - \omega_0 t >> 1$$

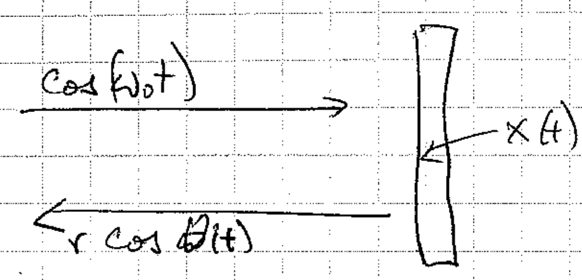
(b) small modulation index $\Longrightarrow$ Bandwidth like AM case

$$\Theta(t) - \omega_0 t << 1$$

If large modulation index, then need to work to do demodulation.

- track phase
- differentiation $\longrightarrow$ $\frac{dy}{dt} = -\sin\Theta \frac{d\Theta}{dt} = -\sin\Theta\,[\omega_0 + \alpha x(t)]$
- ~~Qud~~ Quadrature demodulation (~~xxx~~ low index)

Example



Doppler shift:

When a train coming toward us sounds a whistle, the frequency is shifted up. When it is going away the frequency is shifted down.

This is termed the Doppler effect. Light reflected from a moving surface is Doppler shifted. The amount of the shift is proportional to the velocity

$$\frac{\Delta\omega}{\omega_0} = \frac{v}{c}$$ for small velocities toward the beam

So if we look at $\frac{d\theta}{dt}$, we can get an estimate of the velocity signal

$$\frac{d\theta}{dt} = \omega_0 + \omega_0\left(\frac{v(t)}{c}\right)$$

OK - how much change in frequency?

For red light $\hbar\omega_0 \approx 2\,eV$

$$f_0 \approx 2 \cdot (2.4 \times 10^{14}\,Hz)$$

$$c \approx 3 \times 10^{10}\,\frac{cm}{sec}$$

Suppose audio frequency $\sim 10^3\,Hz$

Distances $\sim 1\,n$

$$v \sim 1nm\,10^3\,Hz \sim 10^{-7}\,cm \cdot 10^3\,sec^{-1}$$

$$\sim 10^{-4}\,\frac{cm}{sec}$$

$$\Delta\omega \sim \omega_0 \cdot \frac{v}{c}$$

$$\sim 2\pi \cdot (4.8 \times 10^{14}\,Hz) \cdot \frac{10^{-4}\,cm/sec}{3 \times 10^{10}\,cm/sec}$$

$$= 2\pi\,(4.8 \times 10^{14})\,\frac{10^{-14}}{3}$$

$$= 10\,Rad/sec \rightarrow 1.6\,Hz$$

Thinking about the Reflected signal

$$r \cos \theta(t) = r \cos\left(\int^t \omega_0 + \frac{v(t')}{c} \omega_0 \, dt' \right)$$

$$= r \cos\left(\omega_0 t + \omega_0 \int^t \frac{v(t')}{c} dt' \right)$$

$$= r \cos(\omega_0 t) \cos\left[\omega_0 \int^t \frac{v(t')}{c} dt' \right]$$

$$- r \sin(\omega_0 t) \sin\left[\omega_0 \int^t \frac{v(t')}{c} dt' \right]$$

$\sin \omega_0 \int^t \frac{v(t')}{c} dt'$ looks small here

$$\approx r \cos(\omega_0 t) - r \sin(\omega_0 t) \sin\left(\omega_0 \int^t \frac{v(t')}{c} dt' \right)$$

$$\approx r \cos(\omega_0 t) - r \sin(\omega_0 t) \left[\omega_0 \int^t \frac{v(t')}{c} dt' \right]$$

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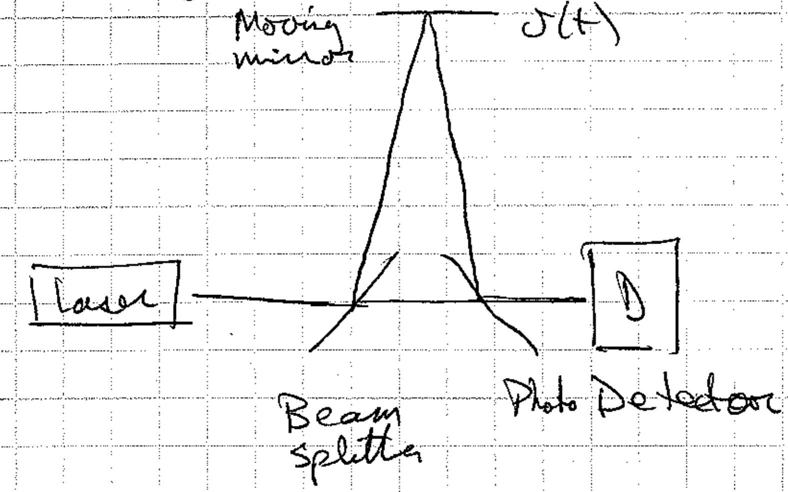
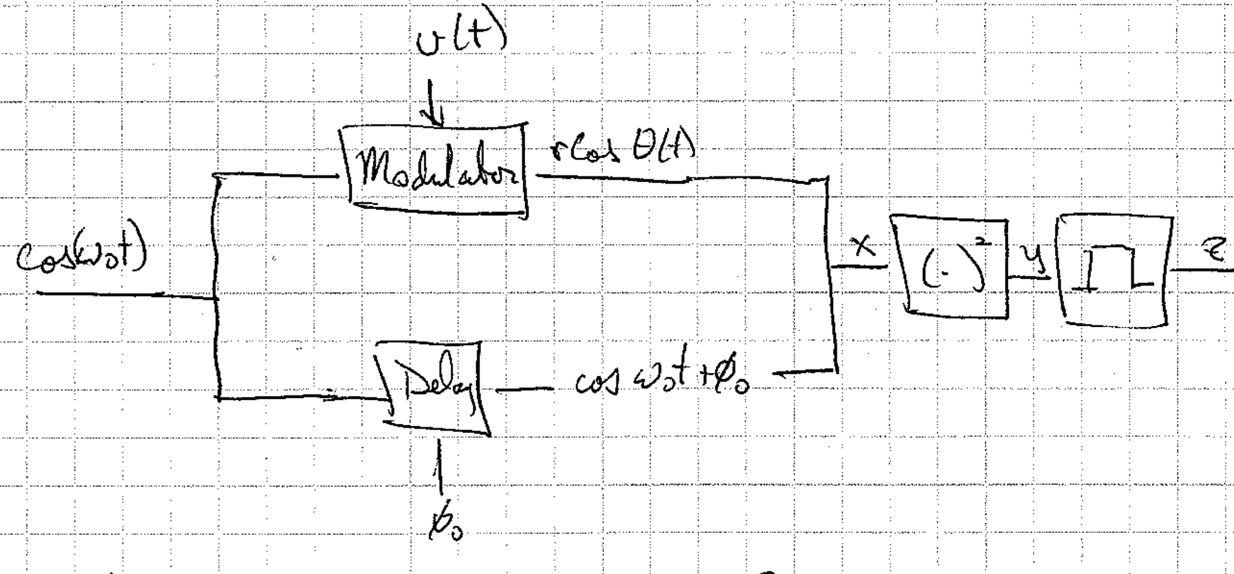
Homodyne detection scheme



Detector is a low pass filter on the square of the signal.

We can make a 6.003-type circuit for it:



$$x(t) = r\cos\theta(t) + \cos[\omega_0 t + \phi_0]$$

$$y(t) = x^2(t) = r^2\cos^2\theta(t) + \cos^2[\omega_0 t + \phi_0] + 2r\cos\theta(t)\cos[\omega_0 t + \phi_0]$$

$$y(t) = r^2\left[\frac{e^{j\theta} + e^{-j\theta}}{2}\right]^2 + r^2\left[\frac{e^{j\omega_0 t+\phi_0} + e^{-j\omega_0 t+\phi_0}}{2}\right]^2$$

$$+ 2r\left[\frac{e^{j\theta} + e^{-j\theta}}{2}\right]\left[\frac{e^{j\omega_0 t+\phi_0} + e^{-j\omega_0 t+\phi_0}}{2}\right]$$

$$= r^2\,\frac{e^{2j\theta} + 2 + e^{-2j\theta}}{4}$$

$$+ r^2\,\frac{e^{2j(\omega_0 t+\phi_0)} + 2 + e^{-2j(\omega_0 t+\phi_0)}}{4}$$

$$+ 2r\,\frac{e^{j(\theta+\omega_0 t+\phi_0)} + e^{j(\theta-\omega_0 t-\phi_0)} + e^{-j(\theta-\omega_0 t-\phi_0)} + e^{-j(\theta+\omega_0 t+\phi_0)}}{4}$$

The low pass filter part of this is

$$z(t) = \frac{r^2}{2} + \frac{1}{2} + \frac{r}{2}\left[e^{j(\theta-\omega_0 t-\phi_0)} + e^{-j(\theta-\omega_0 t-\phi_0)}\right]$$

$$= \frac{1+r^2}{2} + r\cos\left[\theta(t) - \omega_0 t - \phi_0\right]$$ ← Detector sees the difference frequency

Now, if $\phi_0 = \frac{\pi}{2}$, this is

$$z(t) \longrightarrow \frac{r^2+1}{2} + r\sin\left[\theta(t) - \omega_0 t\right]$$

Recall that $\theta(t) = \omega_0 t + \omega_0 \int^t \frac{v(t')}{c}\, dt'$

$$z(t) \longrightarrow \frac{1+r^2}{2} + r \sin\left[\omega_0 \int^t \frac{v(t')}{c}\, dt'\right]$$

$$\approx \frac{1+r^2}{2} + r\, \omega_0 \int^t \frac{v(t')}{c}\, dt'$$

Since $\frac{dx}{dt} = v$ $\quad x =$ position of surface

$$x = \int^t v\, dt'$$

$$z(t) \longrightarrow \frac{1+r^2}{2} + r \frac{\omega_0}{c} x(t)$$

$$\frac{1+r^2}{2} + r \frac{2\pi}{\lambda} x(t)$$

Applications:

Spying – listening to conversations in a Room by bouncing a ~~light of~~ laser beam off of the window

Gravitational wave detector – can see very tiny motions less than $1\,\text{Å}$ easily.

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