6.003R

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I. Laplace Transforms + Z-transforms

Circuits
Differential eqns
Impulse Response
Properties
Convolution

II. Applications

Numerical schemes
2ND order systems
Relaxation systems
Filtering
Feedback

III. Fourier Transforms + Series

Properties
~~Bas~~ Notion of basis expansions
Convolution

IV. Applications

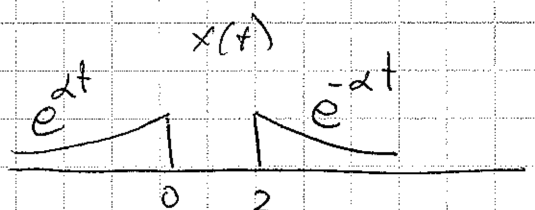
Filtering
Sampling
Modulation + demodulation
Communications systems
~~CD~~ Compact disk

Laplace and Z-transforms

Circuits
Differential + difference eqns
Impulse Response
Inverse transforms
Properties
Convolution

$$X(s) = \int_{-\infty}^{\infty} e^{-st}\, x(t)\, dt$$

Example



$$x(t) = e^{\alpha t} u(-t) + e^{-\alpha t} u(t-2)$$

$$X(t) = \int_{-\infty}^{0} e^{-st} e^{\alpha t}\, dt + \int_{2}^{\infty} e^{-st} e^{-\alpha t}\, dt$$

$$= \left. \frac{e^{(\alpha - s)t}}{\alpha - s} \right|_{-\infty}^{0} + \left. \frac{e^{-(\alpha + s)t}}{-(\alpha + s)} \right|_{2}^{\infty}$$

$$= \frac{1}{\alpha - s} + \frac{e^{-2(\alpha + s)}}{\alpha + s}$$

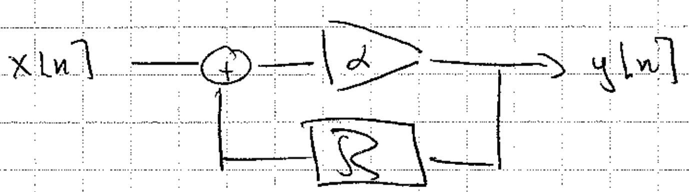
$\mathrm{Re}\{\alpha - s\} > 0$

$\mathrm{Re}\{s\} < \mathrm{Re}\{\alpha\}$

$\mathrm{Re}\{\alpha + s\} > 0$

$\mathrm{Re}\{s\} > \cancel{\alpha}\ \mathrm{Re}\{-\alpha\}$

Example



$$y[n] = \alpha\left(x[n] + y[n-1]\right)$$

$$(1-\alpha R)\, y[n] = \alpha\, x[n]$$

$$y[n] = \frac{\alpha}{1-\alpha R}\, x[n]$$

$$= \alpha\left[1 + \alpha R + (\alpha R)^2 + (\alpha R)^3 + \cdots\right] x[n]$$

$$= \alpha\, x[n] + \alpha^2 x[n-1] + \alpha^3 x[n-2] + \cdots$$

Z-transform:

$$Y(z) = \sum_n z^{-n}\, y[n]$$

$$Z[n] = \sum_n z^{-n}\, y[n-1]$$

$$= \sum_{n'} z^{-(n'+1)}\, y[n']$$

$$= z^{-1} \sum_{n'} z^{-n'}\, y[n']$$

$$= z^{-1}\, Y(z)$$

$$Y(z) = \alpha X(z) + \frac{\alpha}{z}\, Y(z)$$

$$\left(1 - \frac{\alpha}{z}\right) Y(z) = \alpha X(z)$$

$$Y(z) = \frac{\alpha}{1-\frac{\alpha}{z}}\, X(z)$$

$$H(z) = \frac{\alpha}{1 - \frac{\alpha}{z}} = \frac{Y(z)}{X(z)}$$

<u>Impulse Response:</u> $h[n]$

Want $H(z) = \sum_n z^{-n} h[n]$

Try $h[n] = \alpha^{n+1} u[n]$

$$H(z) = \sum_{n=\infty}^{\infty} \frac{\alpha^{n+1}}{z^n} u[n]$$

$$= \alpha \sum_{n=0}^{\infty} \left(\frac{\alpha}{z}\right)^n$$

$$= \alpha \frac{1}{1 - \left(\frac{\alpha}{z}\right)} \qquad \left|\frac{\alpha}{z}\right| < 1$$

$$|z| > |\alpha|$$

<u>General solution</u>

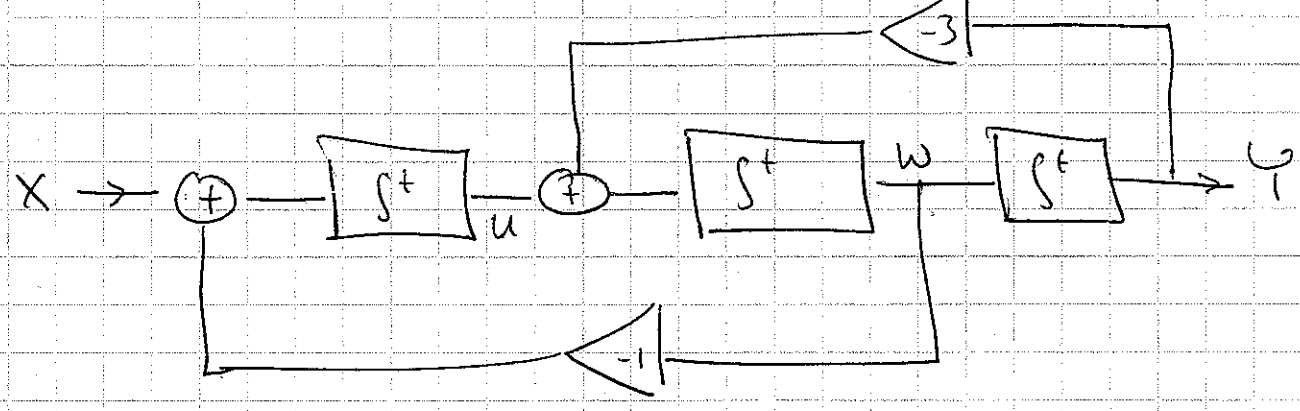
$$Y(z) = H(z) X(z)$$

$$y[n] = h[n] * x[n]$$

$$x[n] = \delta[n] - \delta[n-2]$$

$$y[n] = h[n] - h[n-2] \;\ldots$$

Example



First, define new variables U and W at nodes to help simplify things. Then work out relations

$$Y(s) = \frac{1}{s} W(s)$$

$$W(s) = \frac{1}{s}\left[U(s) - 3Y(s)\right]$$

$$U(s) = \frac{1}{s}\left[X(s) - Y(s)\right]$$

Then eliminate W and U:

$$W(s) = sY(s)$$

$$U(s) = 3Y(s) + sW(s) = (s^2+3)Y(s)$$

End up with constraint

$$(s^2+3)Y(s) = \frac{1}{s}\left[X(s) - Y(s)\right]$$

$$\left[s^3 + 3s + 1\right] Y(s) = X(s)$$

$$\frac{d^3y}{dt^3} + 3\frac{dy}{dt} + y = x(t)$$

Example

$$\bar{X}(s) = \bar{X}(-s)$$

$$x(t) = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \bar{X}(s) e^{st}\, ds$$

$$= \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \bar{X}(-s) e^{st}\, ds$$

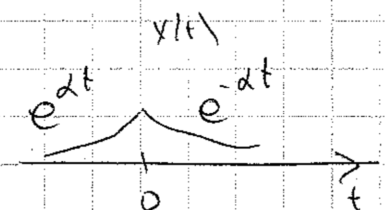
$$s = \sigma + j\omega \qquad s' = -s = -\sigma - j\omega$$

$$ds' = -ds = -\sigma - j d\omega$$

$$x(t) = \frac{1}{2\pi j} \int_{j\infty}^{-j\infty} \bar{X}(s') e^{-s't} (-ds')$$

$$= \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \bar{X}(s') e^{-s't}\, ds' = x(-t)$$

Check it out



$$\bar{X}(s) = \int_{-\infty}^{0} e^{-st} e^{\alpha t}\, dt + \int_{0}^{\infty} e^{-st} e^{-\alpha t}\, dt$$

$$= \left. \frac{e^{(\alpha - s)t}}{\alpha - s} \right|_{-\infty}^{0} + \left. \frac{e^{-(\alpha + s)t}}{-(\alpha + s)} \right|_{0}^{\infty}$$

$$= \frac{1}{\alpha - s} + \frac{1}{\alpha + s}$$

$s < \mathrm{Re}\{\alpha\}$ $\qquad$ $s > \mathrm{Re}\{\alpha\}$

Example

$$\lim_{Re\{s\}\to\infty} \left[s X(s) \right] \to 2$$

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} \, dt$$

let $x(t) \approx X_o u(t)$ for small t

$$\lim_{Re\{s\}\to\infty} X(s) = \int_0^{\infty} X_o e^{-st} \, dt$$

$$\to \frac{1}{s} X_o$$

So, if

$\lim_{Re\{s\}\to\infty} \left[s X(s) \right] \to 2$ then $x(t) \to 2u(t)$ for small t

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Applications:

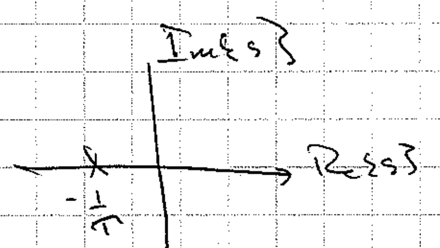
Example 1st order system

$$\frac{dy}{dt} + \frac{y}{\tau} = \frac{x}{\tau}$$

$$sY(s) + \frac{Y(s)}{\tau} = \frac{X(s)}{\tau}$$

$$Y(s)\left(s + \frac{1}{\tau}\right) = \frac{1}{\tau} X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1/\tau}{s + 1/\tau}$$



$$h(t) = \frac{1}{\tau} e^{-t/\tau} u(t)$$

$$H(s) = \int_{-\infty}^{\infty} e^{-st} h(t)\, dt$$

$$= \frac{1}{\tau}\int_{0}^{\infty} e^{-st} e^{-t/\tau}\, dt$$

$$= \frac{1/\tau}{s + 1/\tau} \qquad Re\{s\} > 0$$

Example: second order system

$$h(t) = u(t)\, e^{-t/\tau} \sin(\omega_0 t)$$

$$H(s) = \int_{-\infty}^{\infty} e^{-st}\, h(t)\, dt$$

$$= \int_{-\infty}^{\infty} e^{-st}\, u(t)\, e^{-t/\tau} \sin(\omega_0 t)\, dt$$

$$= \int_{0}^{\infty} e^{-st}\, e^{-t/\tau} \left(\frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j} \right) dt$$

$$= \frac{1}{2j} \left[\frac{1}{s + \frac{1}{\tau} - j\omega_0} - \frac{1}{s + \frac{1}{\tau} + j\omega_0} \right]$$

$$= \frac{1}{2j} \; \frac{s + \frac{1}{\tau} + j\omega_0 \; - \; s + \frac{1}{\tau} - j\omega_0}{\left(s + \frac{1}{\tau}\right)^2 + \omega_0^2}$$

$$= \frac{\omega_0}{\left(s + \frac{1}{\tau}\right)^2 + \omega_0^2}$$

Poles at $\left(s + \frac{1}{\tau}\right)^2 + \omega_0^2 = 0$

$$s + \frac{1}{\tau} = \pm j\omega_0$$

$$s = \pm j\omega_0 - \frac{1}{\tau}$$

Differential eqn

$$Y(s) = H(s)\,X(s)$$

$$Y(s) = \frac{\omega_0}{\left(s+\frac{1}{\tau}\right)^2+\omega_0^2}\,X(s)$$

$$\left[\left(s+\frac{1}{\tau}\right)^2+\omega_0^2\right]Y(s) = \omega_0 X(s)$$

$$\left[s^2+\frac{2s}{\tau}+\frac{1}{\tau^2}+\omega_0^2\right]Y(s) = \omega_0 X(s)$$

$$\left[\frac{d^2}{dt^2}+\frac{2}{\tau}\frac{d}{dt}+\left(\omega_0^2+\frac{1}{\tau^2}\right)\right]y(t) = \omega_0 x(t)$$

Solution

$$y(t) = (x * h)(t)$$

$$y(t) = \int_{-\infty}^{\infty} x(t')\,h(t-t')\,dt'$$

$$= \cancel{C\int_{-\infty}^{\infty} x(t')\,u(t)e^{-t/\tau}\sin\omega_0 t\,dt}$$

$$= \int_{-\infty}^{\infty} x(t')\,u(t-t')\,e^{-(t-t')/\tau}\sin\omega_0(t-t')\,dt'$$

$$= \int_{-\infty}^{t} x(t')\,e^{-(t-t')/\tau}\sin\omega_0(t-t')\,dt'$$

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Example

$$h[n] = u[n]\,\alpha^n \cos(\Omega n)$$

$$H(z) = \sum_n z^{-n} h[n]$$

$$= \sum_{n=-\infty}^{\infty} z^{-n} \alpha^n \cos(\Omega n)\, u[n]$$

$$= \sum_{n=0}^{\infty} \left(\frac{\alpha}{z}\right)^n \left[\frac{e^{j\Omega n} + e^{-j\Omega n}}{2}\right]$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{\alpha e^{j\Omega}}{z}\right)^n + \left(\frac{\alpha e^{-j\Omega}}{z}\right)^n$$

$$= \frac{1}{2}\left[\frac{1}{1-\frac{\alpha e^{j\Omega}}{z}} + \frac{1}{1-\frac{\alpha e^{-j\Omega}}{z}}\right]$$

$$= \frac{1}{2}\left[\frac{z}{z-\alpha e^{j\Omega}} + \frac{z}{z-\alpha e^{-j\Omega}}\right]$$

$$= \frac{1}{2}\left[\frac{z^2 - \alpha z e^{+j\Omega} + z^2 - \alpha z e^{-j\Omega}}{z^2 - \alpha z\left(e^{j\Omega} + e^{-j\Omega}\right) + \alpha^2}\right]$$

$$= \frac{1}{2}\,\frac{2z^2 - 2\alpha z \cos\Omega}{z^2 + \alpha^2 - 2\alpha z \cos\Omega}$$

$$= \frac{z^2 - \alpha z \cos\Omega}{z^2 + \alpha^2 - 2\alpha z \cos\Omega}$$

Poles

$$z = \alpha e^{j\Omega}, \alpha e^{-j\Omega}$$

Difference equ

$$Y(z) = H(z)\,X(z)$$

$$= \frac{z^2 - \alpha z \cos\Omega}{z^2 + \alpha^2 - 2\alpha z \cos\Omega} \, X(z)$$

$$\left[z^2 + \alpha^2 - 2\alpha z \cos\Omega\right] Y(z) = \left(z^2 - \alpha z \cos\Omega\right] X(z)$$

$$R[n] = y[n+2]$$

$$R(z) = \sum_n z^{-n} y[n+2)$$

$$= \sum_{n'} z^{-(n'-2)} y[n']$$

$$= z^2 \sum_{n'} z^{-n'} y[n'] = z^2 Y(z)$$

$$y[n+2] + \alpha^2 y[n] - 2\alpha\cos\Omega\, y[n+1]$$

$$= x[n+2] - \alpha\cos\Omega\, x[n+1]$$

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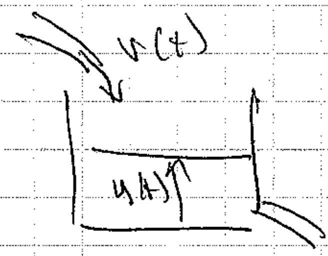
Example



Basic system, no feed back

$$\frac{dy}{dt} + \frac{y}{T} = v(t)$$

$$\left(s + \frac{1}{T}\right) Y(s) = R(s)$$

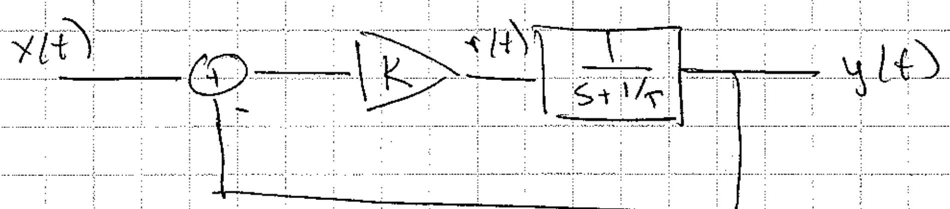
$$H(s) = \frac{Y(s)}{R(s)} = \frac{1}{s + 1/T}$$

Now add feed back

desired

measured

Want $v(t) \approx x(t) - y(t)$



$$v(t) = K(x - y)$$

OK

$$Y(s) = \frac{1}{s+1/\tau} K \left[X(s) - Y(s) \right]$$

$$Y\left(1 + \frac{K}{s+1/\tau}\right) = \frac{K}{s+1/\tau} X$$

$$\frac{Y(s)}{X(s)} = \frac{K/(s+1/\tau)}{1 + K/(s+1/\tau)}$$

$$= \frac{K}{s + 1/\tau + K}$$

Pole is at $s = -\left(K + \frac{1}{\tau}\right)$

if $K >> 1/\tau$, then $\frac{Y(s)}{X(s)} \to 1$ for small s

so it is a controller

$$\left(s + \frac{1}{\tau} + K\right) Y(s) = K X(s)$$

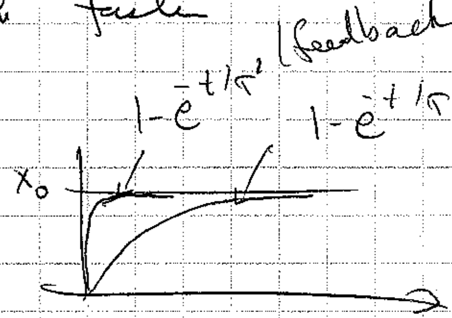
$$\frac{dy}{dt} + \left(\frac{1}{\tau} + K\right) y = Kx$$

$$\cancel{y(t) = \frac{K}{K+1/\tau} x(t)}$$

Impulse response

$$h(t) = K u(t)\, e^{-t/\tau'} \qquad \frac{1}{\tau'} = \frac{1}{\tau} + K$$

Set height, then water goes to that height much faster



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