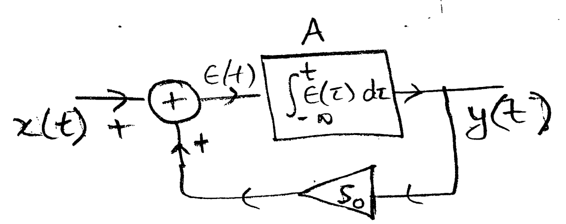
RECITATION #25: REVIEW

4 PARTS TO THE COURSE:

1) OPERATORS, FEEDBACK, DIFFERENTIAL & DIFFERENCE EQUATIONS
2) S AND Z TRANSFORMS
3) FOURIER TRANSFORMS
4) SAMPLING AND MODULATION

Part I



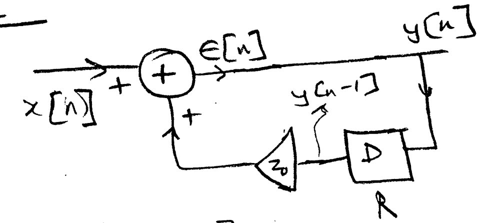
$x(t) = \delta(t)$

$$y(t) = h(t) = \left(\frac{A}{1 - s_0 A}\right)\delta(t) = e^{s_0 t}u(t)$$

$$\frac{dy}{dt} + \frac{y}{\tau} = \frac{x}{\tau}$$

$$h(t) = \frac{1}{\tau}e^{-t/\tau}u(t)$$

$$y(t) = (x * h)(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)\,d\tau$$
$$= (h * x)(t) = \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)\,d\tau$$



$x[n] = \delta[n]$

$$y[n] = h[n] = \left(\frac{1}{1 - z_0 R}\right)\delta[n] = z_0^n u[n]$$

$$y[n] = \alpha y[n] + x[n]$$

$$h[n] = \alpha^n u[n]$$

$$y[n] = (x * h)[n] = \sum_{k=-\infty}^{k=+\infty} x[k]h[n-k]$$
$$(h * x)[n] = \sum_{k=-\infty}^{k=+\infty} h[k]x[n-k]$$

Part II

$$X(s) = \int_{-\infty}^{+\infty} x(t)e^{-st}\,dt \;;\; X(s) = \mathcal{L}\{x(t)\}$$

$$Y(s) = \int_{-\infty}^{+\infty} y(t)e^{-st}\,dt$$

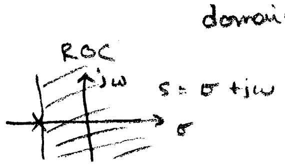
$$H(s) = \int_{-\infty}^{+\infty} h(t)e^{-st}\,dt$$

$e^{st} \to \boxed{H(s)} \to H(s)e^{st}$ ← eigenfunction

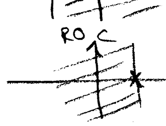
$X(s) \to \boxed{H(s)} \to Y(s) = H(s)X(s)$ ← Convolution turns to multiplication in Laplace domain

$A \equiv \frac{1}{s}$

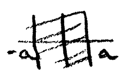
$$\mathcal{L}\{e^{-at}u(t)\} = \frac{1}{s+a}$$



$$\mathcal{L}\{e^{at}u(-t)\} = \frac{1}{a-s}$$



$$\mathcal{L}\{e^{-a|t|}\} = \frac{2a}{a^2 - s^2}$$



$$X(z) = \sum_{n=-\infty}^{n=+\infty} x[n]z^{-n} \;;\; X(z) = \mathcal{Z}\{x[n]\}$$

$$Y(z) = \sum_{n=-\infty}^{n=+\infty} y[n]z^{-n}$$

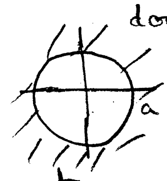
$$H(z) = \sum_{n=-\infty}^{n=+\infty} h[n]z^{-n}$$

$z^n \to \boxed{H(z)} \to H(z)z^n$ ← eigenfunction

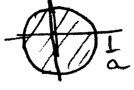
$X(z) \to \boxed{H(z)} \to Y(z) = H(z)X(z)$ ← Convolution turns to multiplication in $\mathcal{Z}$ domain

$R \equiv z^{-1}$

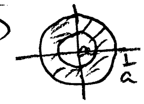
$$\mathcal{Z}\{a^n u[n]\} = \frac{1}{1 - az^{-1}}$$



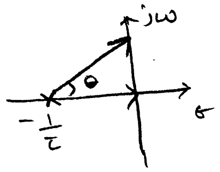
$$\mathcal{Z}\{a^{-n}u[-n-1]\} = \frac{a^{-1}z}{1 - a^{-1}z}$$



$$\mathcal{Z}\{a^{|n|}\} = \frac{1 - a^2}{(1-az)(1-az^{-1})}$$



$s \xrightarrow{e^{sT}} z$



$H(s) = \frac{1}{\tau s+1}$; $H(j\omega) = \frac{1}{\tau j\omega + 1}$; $-\infty < \omega < \infty$

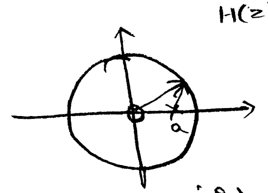
Stable $\Rightarrow$ ROC includes $j\omega$ axis and Fourier transform exists

$\int_{-\infty}^{+\infty} |h(t)|\,dt < \infty$

$\mathcal{L}$: Linearity, Differentiation, Integration, Convolution, Shift, Multiply-by-t, Scaling, Initial Value, Final Value

Causal $\Rightarrow$ $h(t) = 0$; $\forall t < 0$

$H(z) = \frac{1}{1-az^{-1}}$



$H(z) = \frac{1}{1-az^{-1}}$; $H(e^{j\theta}) = \frac{1}{1-ae^{-j\theta}}$; $-\pi < \theta < \pi$

Stable $\Rightarrow$ ROC includes unit circle and Fourier Transform exists.

$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$

Z : Linearity, Differentiation, Integration, Convolution, Shift, Multiply-by-n, Zero Padding, Initial value

Causal $\Rightarrow$ $h[n] = 0$; $\forall n < 0$.

Part III

CTFS
$$\begin{cases} x(t) = x(t-T) \\ x(t) = \sum_{k=-\infty}^{k=+\infty} a_k e^{j2\pi \frac{k}{T} t} \quad \leftarrow \text{synthesis} \\ a_k = \frac{1}{T}\int_{\text{period } T} x(t) e^{-j2\pi\frac{k}{T}t}\,dt \quad \leftarrow \text{analysis} \end{cases}$$

$\downarrow$ periodic extension

CTFT
$$\begin{cases} X(f) = \int_{-\infty}^{+\infty} x(t) e^{-j2\pi f t}\,dt \quad \leftarrow \text{analysis} \\ x(t) = \int_{-\infty}^{+\infty} X(f) e^{j2\pi f t}\,df \quad \leftarrow \text{synthesis} \end{cases}$$

DTFS
$$\begin{cases} x[n] = x[n-N] \\ x[n] = \sum_{k=0}^{k=N-1} a_k e^{j\frac{2\pi}{N}kn} \quad \leftarrow \text{synthesis} \\ a_k = \frac{1}{N}\sum_{n=-\infty}^{n=+\infty} x[n] e^{-j2\pi\frac{k}{N}n} \quad \leftarrow \text{analysis} \end{cases}$$

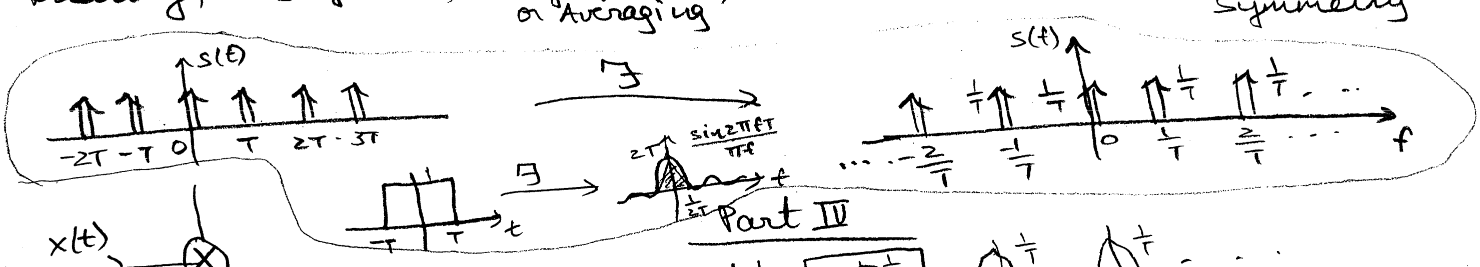
FFT $\approx N\log_2 N$ transform

$\downarrow$ periodic extension

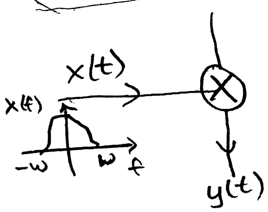
DTFT
$$\begin{cases} X(\theta) = \sum_{n=-\infty}^{n=+\infty} x[n] e^{-j\theta n} \quad \leftarrow \text{analysis} \\ x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\theta) e^{j\theta n}\,d\theta \quad \leftarrow \text{synthesis} \end{cases}$$

$X(\theta) = X(\theta \pm 2n\pi)$

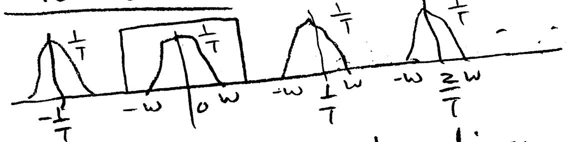
Linearity, Differentiation, Integration, Convolution, Multiply by t, Scaling, Shift
Duality, Conjugation, Asymptotic or Averaging, Parseval's theorem, Even-Odd & Real Symmetry



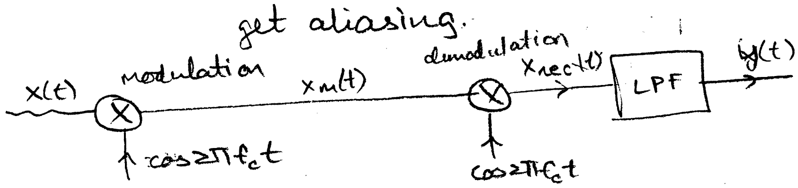
Part IV

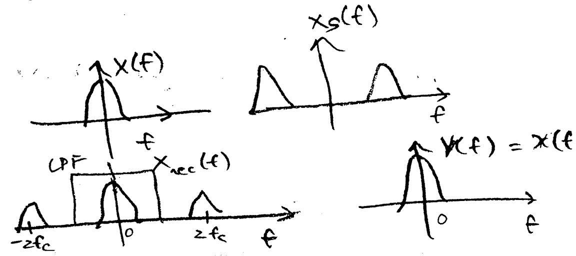


$Y(f) = X(f) * S(f)$ =



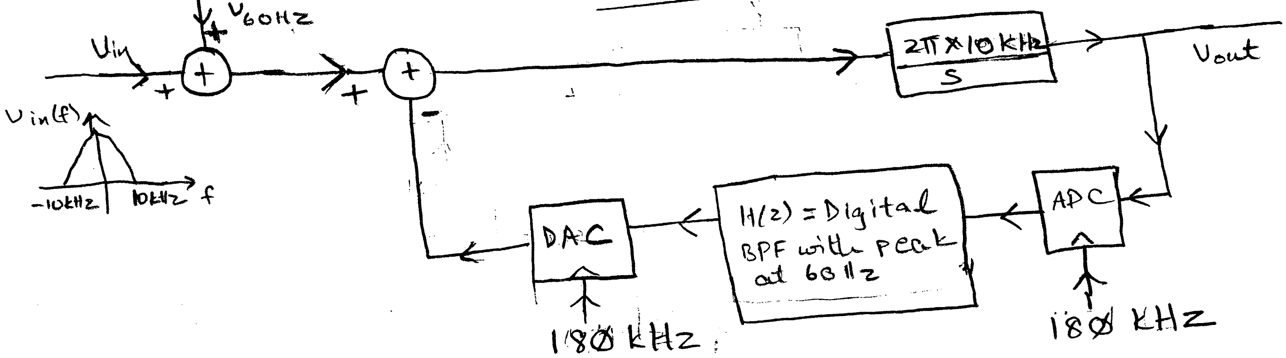
If $w < \frac{1}{2T}$, boxcar sinc filter enables perfect reconstruction. If $w > \frac{1}{2T}$, we get aliasing.



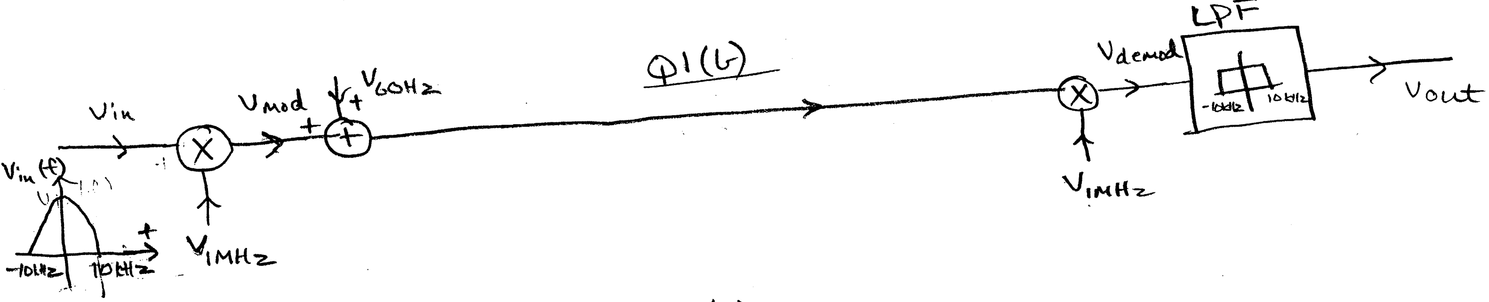


We shall investigate two techniques for the reduction of 60 Hz noise in Figure Q1(a) and Q1(b) respectively.

Q1(a)



Q1(b)



Q1(c)

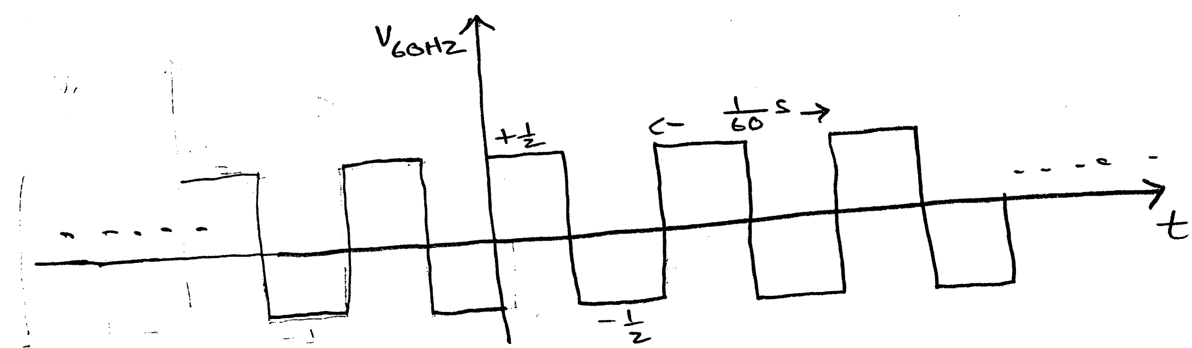
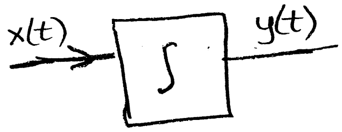


Figure Q1(c) shows an unwanted 60Hz disturbance signal that is periodic and is square-wave in shape with a period of $\frac{1}{60}$ seconds. The two techniques represent two different ways of attenuating the signal energy of V_{60Hz} to a low value at V_{out}. The input signal $V_{in}(t)$ is band limited to 10kHz as shown and identical in Figures Q1(a) and Q1(b). The signal V_{1MHz} is given by $V_{1MHz} = \cos(2\pi \times 10^6 t)$ in Figure Q1(b). The V_{60Hz} signal of Figure Q1(c) represents an identical disturbance in Figures Q1(a) and (

1) From Figure Q1(d) below and the trapezoidal approximation for derivatives, $\frac{x[n]+x[n-1]}{2} = \frac{y[n]-y[n-1]}{T}$,

Q1(d)



$$x(t) = \frac{dy}{dt}$$

$$X(s) = sY(s)$$

where $x[n] = x(nT)$ and $y[n] = y(nT)$ show that a mapping between the s and z planes can be described by

$$s = \frac{2}{T}\left(\frac{z-1}{z+1}\right) \; ; \quad z = \frac{1+\frac{sT}{2}}{1-\frac{sT}{2}}$$

where T is the period of the sampling clock.

2) Map the bandpass filter $H_{bandpass}(s) = \frac{\tau s}{\tau^2 s^2 + \frac{\tau s}{Q} + 1} + 1$

to create H(z) in the feedback path of Figure Q1(a) with $\left(2\pi\tau = \frac{1}{60} \text{ secs}\right)$ and $Q = 100$:

Locate the poles and zeros of $H_{bandpass}(s)$ and map them to corresponding poles and zeros in the z-plane.

3) Draw a Laplace-transform feedback block diagram from Vin(s) to Vout(s) in Figure Q1(a) by representing each ADC & DAC by an appropriate delay element corresponding to their sampling periods and H(z) by an appropriate transfer function. Use Black's formula to

find the transfer function from $V_{in}(s)$ to $V_{out}(s)$.

4) Find $\left| \frac{V_{out}(j2\pi \times 60)}{V_{in}(j2\pi \times 60)} \right|$ by approximating all delay elements as having zero delay

5) Estimate the amplitude and phase of the 60 Hz, 120 Hz, 180 Hz, and 240 Hz harmonics at V_{out} assuming that $V_{in} \equiv 0$ & that V_{60Hz} is the only signal input in Figure Q1(a).

6) In Figure Q1(b), by drawing spectra for $V_{1MHz}(f)$, $V_{in}(f)$, $V_{mod}(f)$, $V_{demod}(f)$, and $V_{out}(f)$, show that V_{out} is a faithful replica of V_{in} if $V_{60Hz} \equiv \emptyset$

7) In Figure Q1(b), if $V_{in} \equiv \emptyset$ and V_{60Hz} is represented by the waveform in Figure Q1(c), estimate the amplitude and phase of the 60 Hz, 120 Hz, 180 Hz, and 240 Hz harmonics at V_{out} if the 10 kHz LPF in Figure Q1(b) is ideal.

8) If the 10 kHz LPF is modeled by

$$H_{LPF}(f) = \frac{1}{1 + j\left(\frac{f}{10^4}\right)},$$ repeat part 7).

9) How would you modify $H(z)$ in Figure Q1(a) to achieve comparable rejection of high-order harmonics as in the architecture of Figure Q1(b)?

SOLUTION OF EXAMPLE PROBLEM

1) $$\frac{x[n]+x[n-1]}{2} = \frac{y[n]-y[n-1]}{T}$$

$$\left(1+z^{-1}\right)\frac{X(z)}{2} = \left(1-z^{-1}\right)\frac{Y(z)}{T}$$

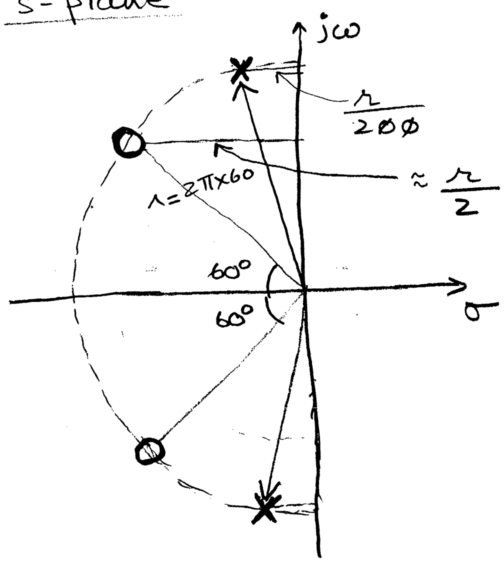
$\dfrac{X(z)}{Y(z)}$ analogous to s & equal to $\dfrac{2}{T}\dfrac{(1-z^{-1})}{(1+z^{-1})} = \dfrac{2}{T}\left(\dfrac{z-1}{z+1}\right)$

$s = \dfrac{2}{T}\left(\dfrac{z-1}{z+1}\right) \Rightarrow z = \dfrac{1+\frac{sT}{2}}{1-\frac{sT}{2}}$ by algebra.

2) $$H_{bandpass}(s) = \frac{\tau s}{\tau^2 s^2 + \frac{\tau s}{\varphi} + 1} + 1 = \frac{\tau^2 s^2 + \tau s\left(1+\frac{1}{\varphi}\right) + 1}{\tau^2 s^2 + \frac{\tau s}{\varphi} + 1}$$

$$= \frac{\tau^2 s^2 + \frac{\tau s}{\left(\frac{\varphi}{\varphi+1}\right)} + 1}{\tau^2 s^2 + \frac{\tau s}{\varphi} + 1}$$

s-plane



$\varphi = 100$

$2\pi\tau = \dfrac{1}{60}$ secs.

zeros at $-2\pi\times 60\left(\frac{1}{2} \pm \frac{\sqrt{3}}{2}j\right)$ in s-plane

poles at $-2\pi\times 60\left(\frac{1}{200} \pm \left(1-\frac{1}{8\times 10^4}\right)j\right) \simeq -2\pi\times 60\left(\frac{1}{200} \pm j\right)$ in s-plan

Sampling $T = (180\,\text{kHz})^{-1}$
zeros in s-plane map to zeros in z-plane according to

$$\frac{1 + \left(-2\pi \times 60\left(\frac{1}{2} \pm \frac{\sqrt{3}}{2}j\right)\right)\frac{1}{180\times 10^3 \times 2}}{1 - \left(-2\pi \times 60\left(\frac{1}{2} \pm \frac{\sqrt{3}}{2}j\right)\right)\frac{1}{180\times 10^3 \times 2}}$$

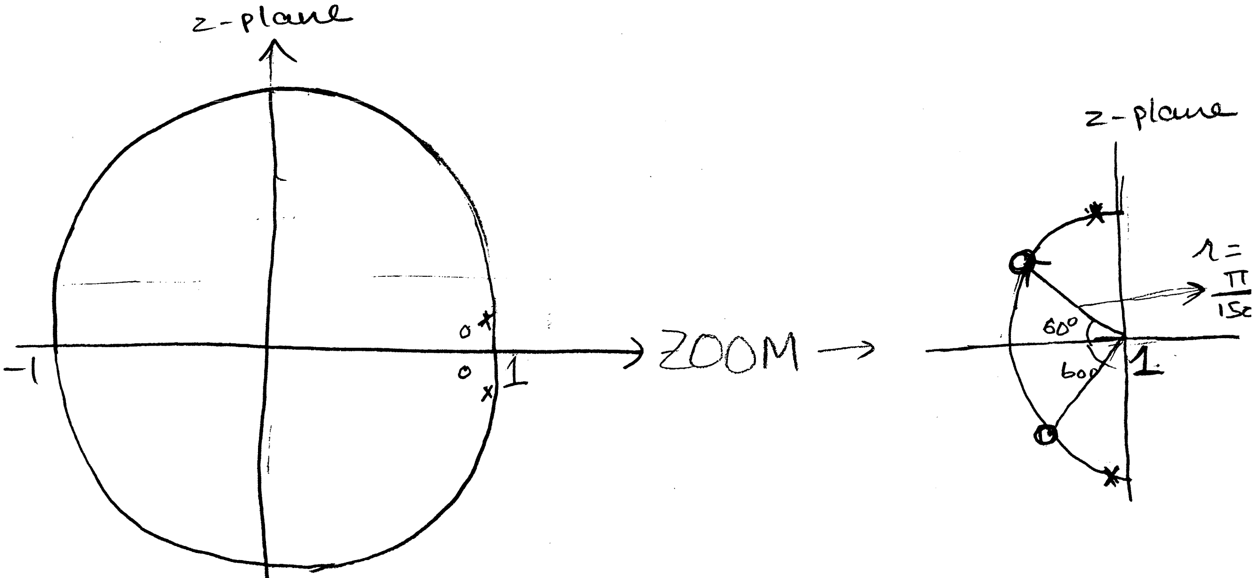
Since denominator is of $1-\epsilon$ form with ϵ small, and numerator is of $1+\epsilon$ form with ϵ small, $\frac{1+\epsilon}{1-\epsilon} \approx 1+2\epsilon$ and we get zeros in the z-plane at

$$1 + \left(\frac{-2\pi \times 60}{180\times 10^3}\right)\left(\frac{1}{2} \pm \frac{\sqrt{3}}{2}j\right) = 1 - \frac{\pi}{1500}\left(\frac{1}{2} \pm \frac{\sqrt{3}}{2}j\right)$$

Similarly, poles are located at

$$1 + \left(\frac{-2\pi \times 60}{180\times 10^3}\right)\left(\frac{1}{200} \pm j\right) = 1 - \frac{\pi}{1500}\left(\frac{1}{200} \pm j\right)$$

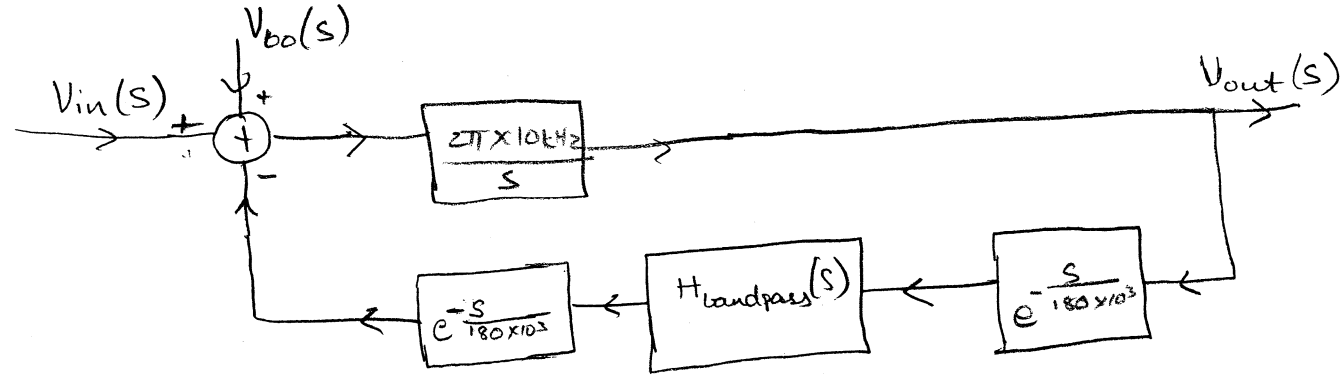
Thus, z-plane poles and zeros look like



splane
$2\pi \times 60 \longrightarrow \frac{\pi}{1500}$ in z-plane and $s = 0 \longrightarrow z = 1$. Pole-zero

plot in s-plane is simply scaled and translated to 1.

3)



$$\frac{V_{out}(s)}{V_{in}(s)} = \frac{\frac{2\pi\times10^4}{s}}{1 + e^{-\frac{s}{90\times10^3}} H_{bandpass}(s)\, \frac{2\pi\times10^4}{s}}$$

4) $$\left|\frac{V_{out}(s)}{V_{in}(s)}\right| = \frac{1}{|H_{bandpass}(s)|}$$ if $L(s)$ is sufficiently large at s, as is true at $s = j\times2\pi\times60$

$$\Big\{ \quad H_{bandpass}(s) \simeq \frac{\left(\cancel{1 - \frac{(2\pi)^2\times60^2}{(2\pi)^2\times60^2}}\right) + j\,\frac{2\pi\times60}{2\pi\times60}\times1.01}{\left(\cancel{1 - \frac{(2\pi)^2\times60^2}{2\pi^2\times60^2}}\right) + j\,\frac{2\pi\times60}{2\pi\times60}\times\frac{1}{100}} = 101$$

$$\frac{2\pi\times10^4}{2\pi\times60} = 160$$

So $L(s) = 160\times101 \approx 16{,}000$

$\Big\}$

Thus $$\boxed{\frac{V_{out}(j2\pi\times60)}{V_{in}(j2\pi\times60)} = \frac{1}{101} \simeq 10^{-2}}$$

5) ~~For~~ ω not near $j2\pi \times 60$,

$$H_{bandpass}(s) = \frac{\tau^2 s^2 + \frac{\tau s}{\Phi(\Phi+1)} + 1}{\tau^2 s^2 + \frac{\tau s}{\Phi} + 1} \approx \frac{\cancel{\tau^2 s^2 + 1}}{\cancel{\tau^2 s^2 + 1}} \simeq 1$$

$$|L(s)| = \left| \frac{2\pi \times 10^4}{s} H_{bandpass}(s) \right| >> 1 \text{ if } s << 2\pi \times 10^4$$

Thus, all other harmonics besides 60 Hz have a transfer function from $V_{60Hz}(s)$ to $V'_{out}(s)$ of 1.

For the square wave, the Fourier series gives

$$a_k = \frac{1}{jk\pi} \quad \forall\, k \text{ odd}$$

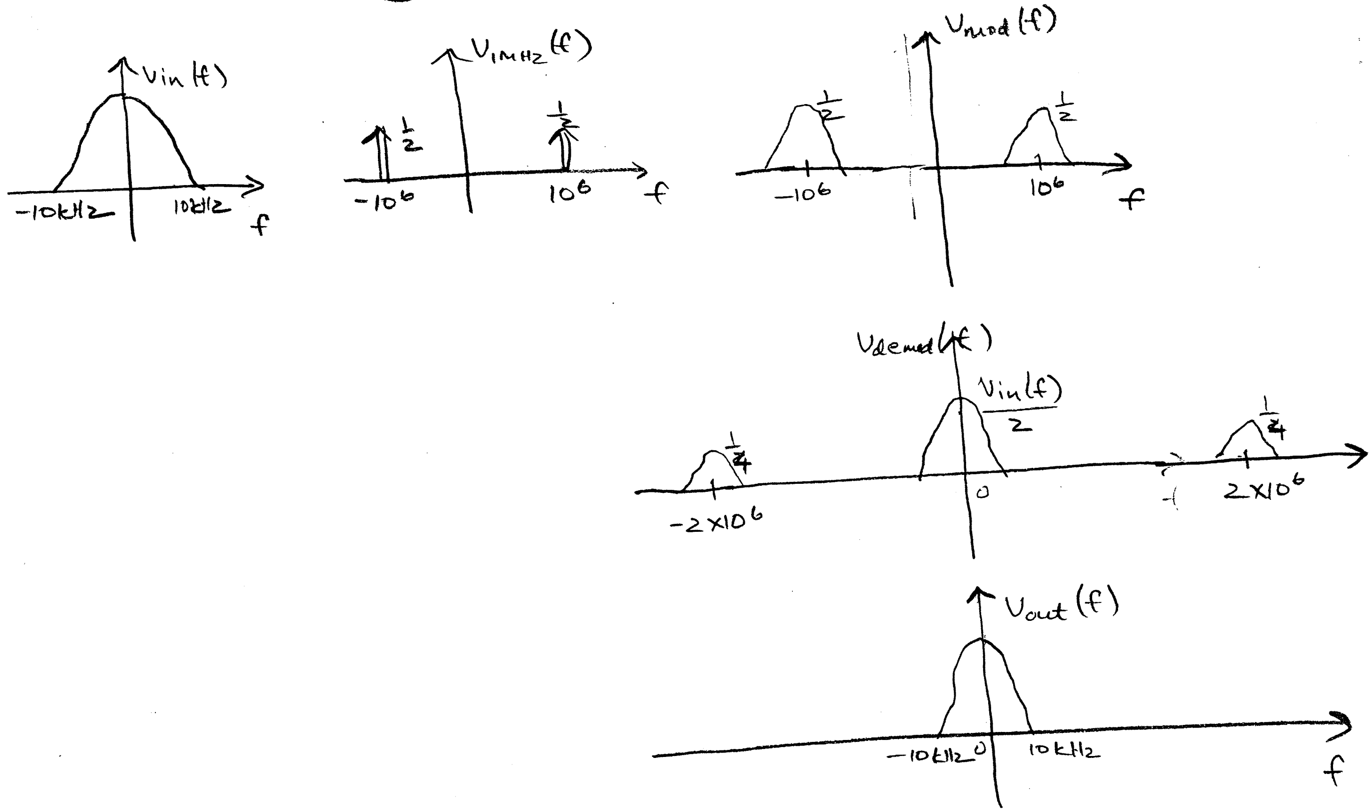
$$= \phi \quad \text{otherwise}$$

Thus, amplitude and phase of various harmonics is given by

$$\boxed{\begin{aligned}
V_{out}(j2\pi \times 60) &= 10^{-2} \times \frac{1}{j \times 1 \times \pi} = 3.2 \times 10^{-3}, \text{ phase is } -90^\circ \\
V_{out}(j2\pi \times 120) &= 1 \times \phi = \phi \quad \text{(no even harmonics)} \\
V_{out}(j2\pi \times 180) &= 1 \times \frac{1}{j \times 3 \times \pi} \simeq 0.1\phi5, \text{ phase is } -90^\circ \\
V_{out}(j2\pi \times 240) &= 1 \times \phi = \phi \quad \text{(no even harmonics)}
\end{aligned}}$$

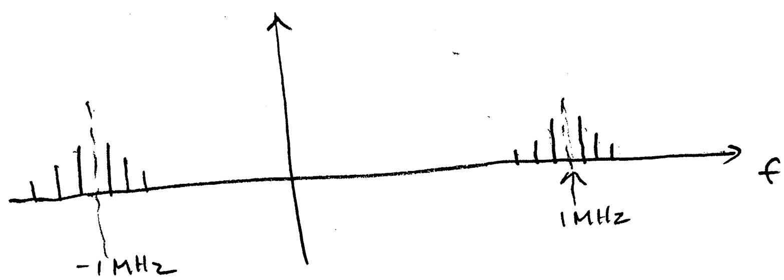
(10)

6)



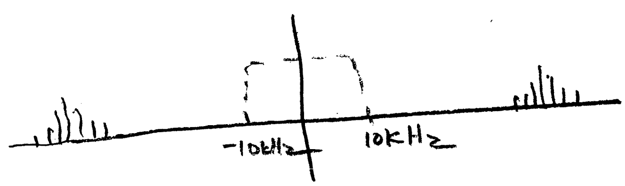
7) $V_{60Hz} \times V_{1MHz} \xrightarrow{\mathcal{F}} V_{60Hz}(f) * V_{1MHz}(f)$

$=$



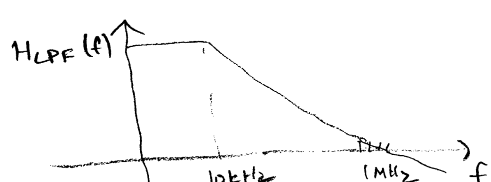
when filtered by a 10KHz LPF

$\boxed{V_{out} = \emptyset}$



for all 60, 120, 180, & 240 Hz harmonics

8)



Attenuation for all odd harmonic is nearly

$$\frac{1}{jk\pi} \times \frac{1}{j\left(\frac{10^6}{10^4}\right)}$$

$$= \frac{-1}{\pi k \times 10^2}$$

$V_{out}(j2\pi\times60) = -3\times10^{-3}$
$V_{out}(j2\pi\times120) = \emptyset$
$V_{out}(j2\pi\times180) = 1\times10^{-3}$
$V_{out}(j2\pi\times240) = \emptyset$

9) i) Increase Q of current pole by $\frac{3.2\times10^{-3}}{3\times10^{-3}} \approx$ i.e to 107

ii) Introduce additional peaks in $H_{bandpass}(s)$ with this higher Q at 180 Hz, 300 Hz, 420 Hz ... etc. and map to corresponding locations in z-plane.

i.e.

$$H_{new}(s) = \left(\sum_{\substack{k=2n+1 \\ n=0}}^{n=\infty} \left(\frac{\tau_k s}{\tau_k^2 s^2 + \frac{\tau_k s}{Q} + 1} \right) \right) + 1$$

$$2\pi \tau_k = \frac{1}{k \times 60}$$