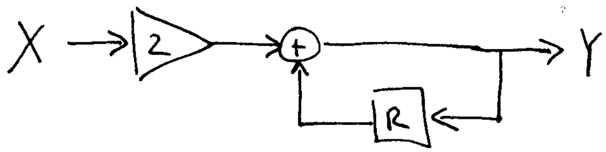
① Given the following difference equation:

$$y[n] = y[n-1] + 2x[n]$$

1. Draw a block diagram for this system.



2. Will this system converge, diverge, or neither?

   unit sample response = step

3. Derive the operator representation for this system. What are the poles?

   $Y = RY + 2X$ $\qquad \frac{Y}{X} = \frac{2}{1-R}$

   Pole at $1 - \frac{1}{z} = 0 \Rightarrow z = 1$

4. If $y[-1] = 41$, and $x[n] = n$, what are the first five outputs of this system?

   $y[0] = 41$ $\qquad y[4] = 53 + 8 = 61$

   $y[1] = 41 + 2 = 43$ $\qquad y[5] = 61 + 10 = 71$

   $y[2] = 43 + 4 = 47$

   $y[3] = 47 + 6 = 53$

5. Write code to generate the system function for this system.

```
sys = sf.SystemFunction(poly.Polynomial[2], poly.Polynomial[-1, 1])
```

6. Write code to implement this system as a state machine.

   2 answers:

```
sysSM = sm.Cascade(sm.Gain(2), sm.FeedbackAdd(sm.Wire(), sm.R(41)))
```

OR

```
= ltism.LTISM([2], [1], None, [41])
```

② How many poles does each of the following systems have? Are the systems stable?

1. $\frac{Y}{X} = \frac{-2 - R + 3R^2}{2R^2 - R}$

$\frac{Y}{X} = \frac{(-2+3R)(-1+R)}{R(-1+2R)}$

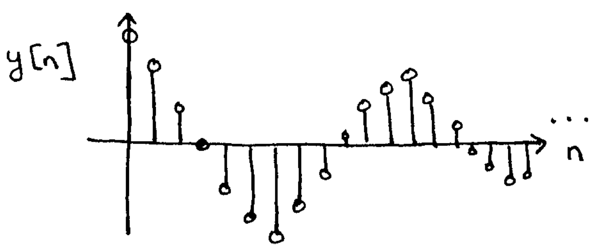
2 poles

2. $y[n] = y[n-1] + y[n-3] + x[n-2] - 2x[n-4]$

$y[n-3]$ is an R^3 term, so 3 poles. Check:

$(-R^3 - R + 1)Y = (R^2 - 2R^4)X$

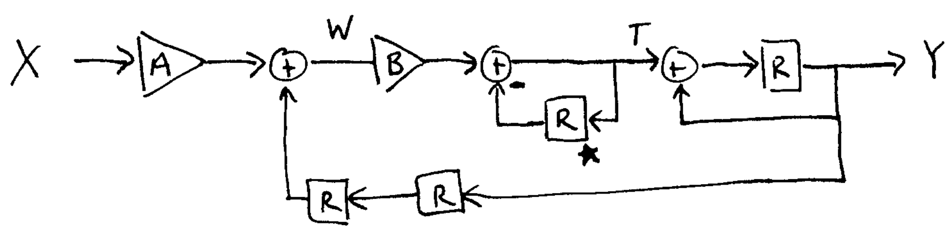
$\frac{Y}{X} = \frac{R^2(1-2R^2)}{(1-R-R^3)}$ No pole/zero cancellation, so 3 poles.

3. 

Sinusoidally varying ⇒ complex poles.
Must come in pairs.
Simple exponential decay suggests

$y[n] = a^n \sin(\omega n) = \frac{e^{(\alpha+j\beta)n} + e^{(\alpha-j\beta)n}}{2}$

⇒ 2 poles

4. 

Label intermediate values:

$W = AX + R^2 Y$

$T = BW - RT \Rightarrow (1+R)T = BW \Rightarrow T = \frac{ABX + BR^2Y}{1+R}$

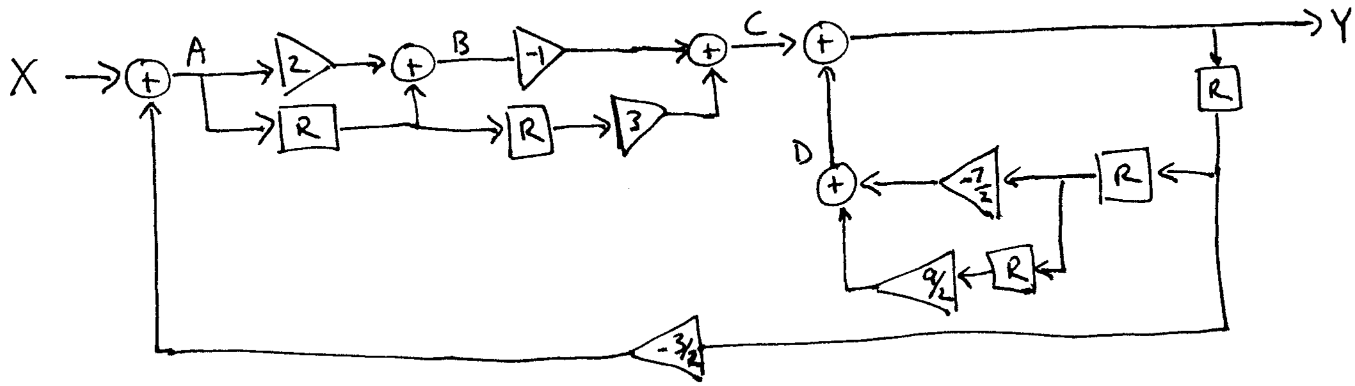
$Y = RY + RT \Rightarrow (1-R)Y = \frac{R}{1+R}(ABX + BR^2Y)$

$(1-R)(1+R)Y = ABRX + BR^3Y \Rightarrow \frac{Y}{X} = \frac{ABR}{-BR^3 - R^2 + 1}$ 3 poles.

4a. How would the answer change if we moved the delay marked by ★ into the forward path?
The 2nd eq would have BRW as the 1st term, so we'd get 4 poles.

4b. For the original system, how many poles can be real? 1 or 3 (since 0 or 2 can be complex)

(3) Simplify this block diagram:



Label intermediate values:

$A = X - \frac{3}{2} RY$

$B = (2+R)A$

$C = 3R^2 A - B = (3R^2 - R - 2)A$

$D = -\frac{7}{2} R^2 Y + \frac{9}{2} R^3 Y$

$$Y = C + D = (3R^2 - R - 2)(X - \tfrac{3}{2} RY) - \tfrac{7}{2} R^2 Y + \tfrac{9}{2} R^3 Y$$

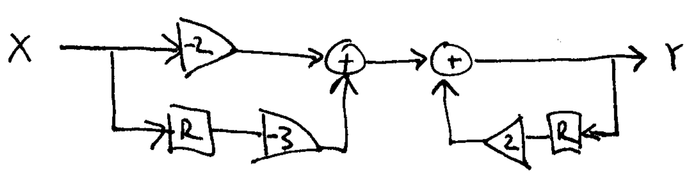
$$= (3R^2 - R - 2)X - \tfrac{9}{2} R^3 Y + \tfrac{3}{2} R^2 Y + 3RY - \tfrac{7}{2} R^2 Y + \tfrac{9}{2} R^3 Y$$

$$= (3R^2 - R - 2)X + (-2R^2 + 3R)Y$$

$$(1 - 3R + 2R^2)Y = (-2 - R + 3R^2)X$$

$$\frac{Y}{X} = \frac{-2 - R + 3R^2}{1 - 3R + 2R^2} = \frac{(2 + 3R)(-1 + R)}{(-1 + 2R)(-1 + R)} = \frac{2 + 3R}{-1 + 2R} = \frac{-2 - 3R}{1 - 2R}$$

$$Y = -2X - 3RX + 2RY$$



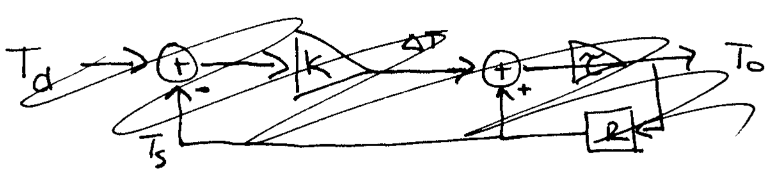
(4) Thermoelectric (a.k.a. Peltier) devices can transfer heat in one of two directions based on the direction of current flow into/out-of the device.

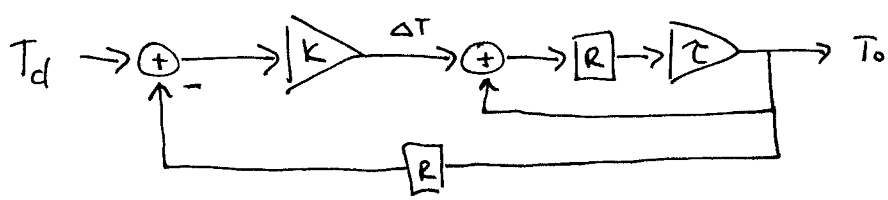
FridgOven Inc., an MIT-based startup, wants to use these devices to make a kitchen appliance that will keep a pre-made dinner cool during the day, then heat it up automatically in the evening.

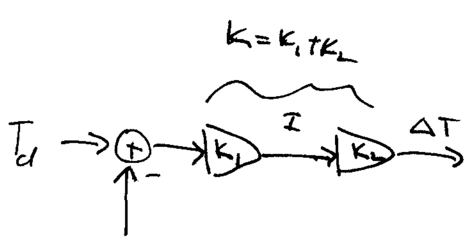
As a recent 6.01 graduate, you've been hired to design the control system for this appliance. You are told the following:

1. The input is a desired temperature $T_d[n]$.
2. The rate of temperature change in the appliance is proportional to the rate of heat transfer, which in turn is proportional to the current supplied to the Peltier.
3. A thermometer reports the average temperature in the appliance over the previous minute (i.e., approximately the temperature 30 seconds ago).

1. Draw a block diagram for this system.







2. Derive $\frac{T_o}{T_d}$, where T_o is the temperature of the appliance.

$$\cancel{\Delta T = k(T_d - T_s) = k(T_d - RT_o)}$$

$$T_o \cancel{= \tau(RT_o + \Delta T) = kT_d + (1-k)RT_o}$$

$$= \cancel{\tau R T_o + \tau k T_d - \tau k R T_o = \tau R(1-k)T_o + \tau k T_d}$$

$$\Delta T = k(T_d - T_s) = k(T_d - RT_o)$$

$$T_o = \tau R(\Delta T + T_o) = \tau R \Delta T + \tau R T_o = \tau k R T_d - \tau k R^2 T_o + \tau R T_o$$

$$T_o(1 - \tau R + \tau k R^2) = \tau k R T_d \qquad \frac{T_o}{T_d} = \frac{\tau k R}{1 - \tau R + \tau k R^2}$$

3. What are the poles of this system? Plot them as a function of gain.

Roots of $1 - \frac{\tau}{z} + \frac{\tau k}{z^2}$, or $z^2 - \tau z + \tau k = 0$

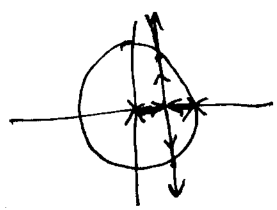
$$z = \frac{\tau \pm \sqrt{\tau^2 - 4\tau k}}{2} \qquad \text{Let } \tau = 1 \text{ time step.}$$

$$z = \frac{1 \pm \sqrt{1-4k}}{2}$$

$k=0$, poles are at 0, 1.

$k = 1/4$, poles are at $1/2$, $1/2$.

$1/4 < k$, poles are complex.



4. For what gains is this system stable? For what gains are the poles real?

Stable for $|1 + \sqrt{1-4k}| \leq 2$, i.e., $1 - 4k \geq -3$

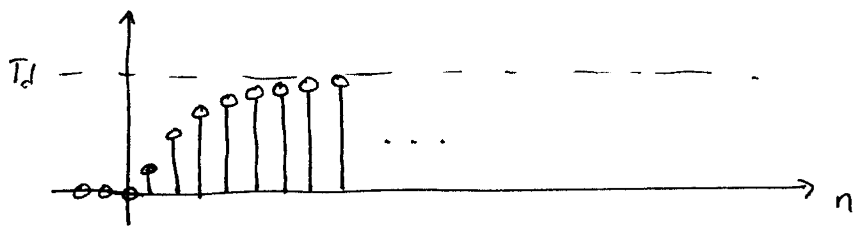
$4k \leq 4$, $k \leq 1$.

Real for $k \leq 1/4$.

5. Plot the time response of this system for a step increase in T_d for the largest gain for which poles are real.

$$\frac{T_o}{T_d} = \frac{KR}{1-R+KR^2} \qquad T_o[n] - T_o[n-1] + \frac{1}{4} T_o[n-2] = \frac{1}{4} T_d[n-1]$$

$$T_o[n] = T_o[n-1] + \frac{1}{4}\left(T_d[n-1] - T_o[n-2]\right)$$



6. Plot the time response to the same input for twice the gain used in part 5.

$$T_o[n] = T_o[n-1] + \frac{1}{2}\left(T_d[n-1] - T_o[n-2]\right)$$