6.014 Lecture 21:
Magnetic Forces and Devices

A. Overview

Magnetic forces are central to a wide array of actuators and sensors. These forces can be calculated using either energy methods or the Lorentz force law, which is:

$$\overline{f} = q(\overline{E} + \overline{v} \times \mu_o \overline{H}) \text{ Newtons} \tag{1}$$

where q is the charge (Coulombs) on which the force acts, and $\overline{v}$ is the charge velocity. The energy method computes the force $\overline{f}$ needed to move an object in the z direction by differentiating the total system energy w_T with respect to displacement z:

$$\overline{f} = dw_T/dz \tag{2}$$

Both these methods are illustrated below, first for the simple case of electrons moving in vacuum, and then for forces on wires and in motors and generators. Finally the Hall effect is discussed.

B. Lorentz Forces on Charges in Vacuum

Consider the evacuated tube illustrated in Figure L21-1a where electrons with charge –e are boiled off from the heated cathode and are accelerated toward the anode with velocity $\overline{v}$ by the electric field $\overline{E}$, which is produced by the voltage V between anode and cathode. The magnetic Lorentz force on the charge –e (-1.6021×10^{19} C) is easily found from (1) to be:

$$\overline{f} = -e\, \overline{v} \times \mu_o \overline{H} \text{ [N]} \tag{3}$$

In high-voltage cathode-ray tubes (CRT's) used for television or computer displays the electrons are quickly accelerated near the cathode and then move with energy ~eV [J], where V is the tube voltage (see (3) in the notes for Lecture 20). Since:

$$eV = mv^2/2 \tag{4}$$

it follows that $v = (2eV/m)^{0.5}$, where m is the electron mass (9.107×10^{-31} kg). Thus electrons moving in a CRT such as that illustrated in Figure L21-1b are deflected upwards, where $\overline{H}$ is directed out of the paper and the electrons move toward the right; the magnitude of the force on the electron is $ev\mu_oH$ [N].

In the special case where V and $\overline{E}$ are zero, a free electron moving perpendicular to a magnetic field $\overline{B}$ will experience a force $\overline{f}$ orthogonal to its velocity vector $\overline{v}$, as illustrated in Figure 21-1c. Since this force $|\overline{f}|$ is always orthogonal to $\overline{v}$, the trajectory of the electron will be circular (radius R) at angular frequency ω_e [radians s^{-1}]:

$$|f| = ev\mu_o H = m_e a = m_e \omega_e^2 R = m_e v \omega_e \quad (5)$$

where $v = \omega_e R$. We can solve (5) for this "electron cyclotron frequency" ω_e:

$$\omega_e = e\mu_o H/m_e \quad (6)$$

which is independent of v and the electron energy (provided the electron is not relativistic). Thus magnetic field strengths H can be measured by observing the radiation frequency ω_e of free electrons in the region of interest.

C. Magnetic Forces on Currents in Wires

The Lorentz force law can also be used to compute forces on electrons in wires. If there is no net charge and no current flowing in a wire, the forces on the positive and negative charges all cancel. However, if n carriers per meter of charge q are flowing in a wire, as illustrated in Figure L21-2a, then the total force per meter on the wire is:

$$\overline{F} = nq\,\overline{v} \times \mu_o\,\overline{H} = \overline{I} \times \mu_o\,\overline{H}\ \ [Nm^{-1}] \quad (7)$$

where $\overline{I}$ is the current vector for the wire ($I = nq\,\overline{v}$).

Consider two parallel wires carrying the same current I in the same direction and separated by distance r, as illustrated in Figure L21-2b. We can easily find $\overline{H}(r)$ from Ampere's law and the cylindrical symmetry:

$$\int_C \overline{H} \bullet d\,\overline{s} = I = 2\pi r H \ \Rightarrow\ H = I/2\pi r \quad (8)$$

The mutual force $\overline{f}$ attracting the two parallel wires is then found from (7) and (8) to be:

$$|F| = \mu_o I^2/2\pi r\ [Nm^{-1}] \quad (9)$$

The simplicity of this equation and the ease of measurement of F, I, and r led to its use in definition of a Henry and the value for the permeability of free space, $\mu_o = 4\pi\times10^{-7}$ henries/meter. If the two currents are in opposite directions, the force is repulsive. For example, if I = 10 amperes and r = 2 millimeters, then (9) yields $F = 4\pi\times10^{-7} \times 10^2/2\pi2\times10^{-3} = 0.01$ Newtons/meter; this is approximately the average repulsive force between the two wires in a 120-volt AC lamp cord delivering one kilowatt. These forces are attractive when the currents are parallel, so if we consider a wire of four quadrants, as illustrated in Figure L21-2c, they will squeeze together due to the "pinch effect". At extreme currents, these forces can actually crush a wire. The same effect can pinch electron beams flowing in charge-neutral plasmas.

D. Voltage Generation by Motion Across `H

Wires moving across magnetic field lines acquire an open-circuit voltage that follows directly from the Lorentz force law (1). Consider the electron illustrated in Figure L21-3a, which has charge –e and velocity $\overline{v}$. It is moving perpendicular to $\overline{H}$ and therefore experiences a force on it of $-e\ \overline{v} \times \mu_o\ \overline{H}$. It experiences that force even inside a moving wire and will accelerate in response to it. If an open-circuited wire of length W moves at $\overline{v}$ perpendicular to $\overline{H}$, all the electrons will move until equilibrium is reached for which the net force on any electron is zero. Otherwise there would be no equilibrium. The balancing force inside a wire is electric and is equal to $-e\ \overline{E}$, as illustrated. That is, the charges will move inside the wire and accumulate until there is sufficient electric potential across the wire to halt their movement. Specifically, this Lorentz force balance requires:

$$-e\ \overline{v} \times \mu_o\ \overline{H} = e\ \overline{E} \tag{10}$$

$$\overline{E} = -\ \overline{v} \times \mu_o\ \overline{H} \tag{11}$$

Figure L21-3b illustrates such a wire of length W moving at $\overline{v}$ perpendicular to $\overline{H}$. If the wire were open-circuited, the potential Φ across it would be the integral of the electric field necessary to cancel the magnetic forces on the electrons, where:

$$\Phi = v\mu_o HW \tag{12}$$

and the signs and directions are as indicated in the figure.

If the moving wire were then connected to a circuit, as illustrated, a current I could flow, depending on Φ and the Thevenin equivalent circuit elements V and R.

The current I induced by wire motion is governed by Ohm's law:

$$I = (V - \Phi)/R \tag{13}$$

which can be positive or negative, depending on the relative values of V and Φ. From (7) we see that the associated total force $\overline{f}$ exerted on the wire by $\overline{H}$ is:

$$\overline{f} = \overline{I} \times \mu_o\ \overline{H}W = \hat{x}\,\mu_o HW(V - \Phi)/R \tag{14}$$

where the unit vector $\hat{x}$ is parallel to $\overline{v}$.

Equation (14) enables us to compute the mechanical power delivered to or by the wire. If the voltage source V is strong enough, then the system functions as a motor and the mechanical power delivered to the environment by the wire is:

$$P_m = \overline{f} \bullet \overline{v} = v\mu_o HW(V - \Phi)/R = \Phi(V - \Phi)/R \ \ [W] \tag{15}$$

The electrical power delivered by the moving wire to the battery and resistor is:

$$\begin{aligned} P_e &= -VI + I^2R = -V(V - \Phi)/R + (V - \Phi)^2/R = [(V - \Phi)/R][-V + (V - \Phi)] \\ &= -\Phi\,(V - \Phi)/R \ \ [W] \end{aligned} \tag{16}$$

This equals the mechanical power P_m of (15) delivered to the wire. If V is zero, then the wire delivers maximum electrical power, Φ^2/R. As V increases, this delivered power diminishes and then becomes negative as the system ceases to be a generator and becomes a motor. As a motor the mechanical power delivered to the wire by the environment becomes negative, and the electrical power delivered by the battery becomes positive. That is, we have a:

Motor: If mechanical power out > 0

$$\text{If } V > \Phi = v\mu_o HW \tag{17}$$

Generator: If electrical power out > 0

$$\text{If } V < \Phi, \text{ or } v > V/\mu_o HW \tag{18}$$

We call Φ the "back voltage" of a motor and when it exceeds the voltage V of the power source, it supplies power to it, becoming a generator. This occurs as the velocity v increases above the threshold given by (18). When $V = \Phi$, the motor moves freely without any electromechanical forces.

E. Rotary Wire Motors

For efficient continuous electromechanical power conversion, whether as a motor or generator, rotary devices are most efficient because the motion is continuous. Figure L21-5a illustrates a motor comprising a single loop of wire carrying current I in the uniform magnetic field $\overline{H}$. The total torque on the motor axle is found by adding the contributions from each of the four sides of the current loop; only the longitudinal elements of length W at radius r contribute. The total torque T = fr corresponding to the forces given by (7) is thus:

$$T = 2I\mu_o HWr \ \ [Nm] \tag{19}$$

Because the field H is uniform, the torque goes to zero only when the wire loop is vertical and temporarily no field lines are being cut due to rotor motion. This torque history is plotted in Figure L21-5b, and goes both positive and negative, averaging to zero. In order to achieve a non-zero average torque, a commutator can be added, as suggested in Figure L21-5c. The two carbon brushes pick up charge from the rotating commutator contacts that pass the current I to the external environment. By reversing the direction of the motor current twice per revolution, the more nearly constant torque history illustrated by the dashed line in Figure L21-5b is obtained and power conversion is maximized.

An example of a two-pole commutated motor is shown in Figure L21-6a. We assume: 1) $B = \mu_o H = 1$ Tesla (10,000 gauss), provided by permanent magnets in the stator, 2) that the rotor has one 100-turn coil (N = 100) of area $WR = A = 10^{-3}$ m^2, and 3) the perfectly commutated motor is driven by a 24-volt DC power supply. The maximum driven angular frequency ω_{max} is obtained if this motor is unloaded and $\Phi_{max} = V = 24$ volts. Using (12), the back-voltage Φ_{max} produced by the wires moving at velocity $v = \omega R$ past the magnetic field H is:

$$\Phi_{max} = 2Nv\mu_o HW = NA\mu_o H\omega_{max} = 24 \tag{20}$$

so it follows from (20) that:

$$\omega_{max} = \Phi_{max}/NA\mu_o H = 24/(100\times10^{-3}\times1) = 240 \text{ rs}^{-1} \Rightarrow \text{ ~2300 rpm} \tag{21}$$

More typical values for B are ~0.1 (lower by a factor of 10), leading to ω_{max} of ~23,000 rpm maximum. Although this maximum speed is ten times greater, the motor torque and power at any given ω are reduced by a factor of ten. The maximum torque T is obtained for this motor when the current is maximum:

$$T = NI\mu_o HA = 100\ I \times 10^{-3}\ \text{[Nm]} \quad (\rightarrow \infty \text{ if } I \rightarrow \infty) \tag{22}$$

Normally the current I is limited by the power supply to some value, say 10 A. Then T_{max} for this motor would be 1 Nm (equivalent to a force of 100 Newtons at a radius of 1 cm).

F. Rotary Motor Torque/Power/Speed Relations

The mechanical power output from a motor equals $T\omega$ [Nms^{-1}], where T is a function of I, as given by (22), I is a function of Φ (see (13)), and Φ is proportional to ω. As a result, the mechanical power output of a motor such as that illustrated in Figure L21-7a is:

$$P_m = \omega T = \omega NI\mu_o HA \tag{23}$$

Substituting $I = (V - \Phi)/R$ and $\Phi = NA\mu_o H\omega$ yields:

$$P_m = \omega N(V - NA\mu_o H\omega)\ \mu_o HA/R \tag{24}$$

This mechanical power P_m delivered is a function of ω and peaks at P_{max} when $\omega = \omega_p$, as illustrated in Figure L21-7b.

We can solve for ω_p by setting $dP_m/d\omega$ to zero using (24); this yields:

$$\omega_p = V/2NA\mu_oH = \omega_{max}/2 \qquad (25)$$

where ω_{max} was given by (21) and $\Phi_{max}= V$. At this frequency ω_p the mechanical power out (24) peaks at P_p:

$$P_p = \omega_pT = (V/2NA\mu_oH)N(V - [NA\mu_oH\ V/2NA\mu_oH])\mu_oHA\ /R = V^2/4R \qquad (26)$$

where most of the terms cancel. This same peak power would be dissipated if the motor were simply replaced by a resistor of value R, where R is the same as the Thevenin source resistance. That is, the maximum power is converted to mechanical work when the motor looks to the Thevenin source like a matched load R, and the source voltage V is divided equally across the source resistance R and the load "R".

G. Rotary Generator Power/Speed Relations

Similar results are obtained when the motor of Figure L21-6 is used as a generator by connecting the commutated windings to a resistive load R and forcefully turning the rotor. The output current $I = \Phi/R$ is then a function of ω, where:

$$\Phi = NA\mu_oH\omega \qquad (27)$$

and the electrical power P_e delivered to the load R is:

$$P_e = \Phi I = \Phi^2/R = (\omega NA\mu_oH)^2/R \qquad (28)$$

To maximize P_e we simply maximize ω, although both the current and voltage out must be limited to prevent electrical breakdown or melting of the motor electrical insulation. Some motors are used only for bursts of output power, and the temporary heating of the wires can be tolerated if the heat capacity of the motor is sufficient. Also, rotors may fly apart if ω is sufficiently high. Generally the tip speed of the rotor is kept below the speed of sound for this reason and to reduce air drag.

The Thevenin source resistance R_s of an electrical generator is simply the dissipative resistance in the coil because the Thevenin voltage Φ does not depend on the current I; any change in the generator output voltage at constant ω is due only to that resistance.

H. Hall Effect Sensors

Any conductor conveys current I by means of charged carriers, which are electrons in metals, and include both electrons and holes in semiconductors. Figure L21-8b illustrates such a material for which the slab width is W and a magnetic field $\overline{H}$ is imposed perpendicular to both the current I and the slab width W. Carriers (n per m) with charge q drifting at velocity v in the direction of I conveys current I = nqv and experience an effective lateral Hall electric field $\overline{E}_H = -\ \overline{v} \times \mu_o\ \overline{H}$ [Vm^{-1}] across the width of the slab, as given by (11). The total Hall voltage V_H is:

$$V_H = WE_H = -Wv\mu_o H \tag{29}$$

so the carrier drift velocity v can be determined:

$$v = -V_H/W\mu_o H \ \ [ms^{-1}] \tag{30}$$

Once v is determined, the product of carrier density and carrier charge can be determined:

$$nq = I/v \tag{31}$$

Conversely, when the natural value for v is known, Hall effect sensors are commonly used to measure H using (29).