MAGNETIC FORCES ON CHARGES

Lorentz Force Law:
\[ \mathbf{f} = q(\mathbf{E} + \mathbf{v} \times \mathbf{\mu}_0 \mathbf{H}) \] Newtons

\( q = \) electric charge (Coulombs)
\( \mathbf{v} = \) velocity vector (m s\(^{-1}\))

Example: magnetic forces on an electron beam:

Lateral forces:
\[ \mathbf{f} = q\mathbf{v} \times \mathbf{\mu}_0 \mathbf{H} \]

Heater filament
Cathode
Anode

Electrostatic deflection best for small \( \mathbf{v} \),
magnetic deflection best for large \( \mathbf{v} \).

Cyclotron motion:
\[ |\mathbf{f}| = e v \mathbf{\mu}_0 \mathbf{H} = m a = m_e \omega_e^2 R = m_e v \omega_e, \] where \( \mathbf{v} = \omega_e R \)
\[ \omega_e = e \mathbf{\mu}_0 \mathbf{H}/m_e \] "Electron cyclotron frequency" [rs\(^{-1}\)]
\[ \text{e.g. } \omega = 1.6 \times 10^{-19} \times 0.1/9.1 \times 10^{-31} \approx 2.8 \text{ GHz} \] (~MRI)

MAGNETIC FORCES ON CURRENTS IN WIRES

Force equation:
\[ \mathbf{f} = q(\mathbf{E} + \mathbf{v} \times \mathbf{\mu}_0 \mathbf{H}) \] Newtons

Force on wire [Nm\(^{-1}\)]:
\[ \mathbf{f} = n q \mathbf{v} \times \mathbf{\mu}_0 \mathbf{H} = I \times \mathbf{\mu}_0 \mathbf{H}, \] where
\( n \) is the number of conduction electrons m\(^{-1}\)
at \( \mathbf{v} \) and \( I \) is the current vector = \( n q \mathbf{v} \)

Force attracting parallel wires:
\[ |\mathbf{f}| = I \mathbf{\mu}_0 \mathbf{H} = \mu_0 I^2/2\pi r = 2I^2 \times 10^{-7}/r \quad [\text{Nm}^{-1}] \quad (\mu_0 = 4\pi \times 10^{-7}) \]
\[ \oint_{\partial s} \mathbf{H} \cdot d\mathbf{s} = I = 2\pi r H \Rightarrow H = I/2\pi r \]

Example: \( |\mathbf{f}| = 2 \times 10^2 \times 10^{-7} / 0.2 = 0.2[N], \) attract or repulse

Pinch effect:
This experiment defines \( \mu_0 \) and predicts “pinch effect”
The four quadrants of a wire squeeze together \( \propto I^2/r \) and can crush wire, limiting the maximum achievable current density
VOLTAGES PRODUCED BY MOTION ACROSS $\mathbf{R}$

Consider electron inside moving wire:

- Force on that electron: $\mathbf{f}_e = -e(\mathbf{E}_e + \mathbf{v} \times \mathbf{\mu}_0 \mathbf{H})$

For open-circuit wire:

- Force balance: 
  $$\mathbf{f}_e = -e(\mathbf{E}_e + \mathbf{v} \times \mathbf{\mu}_0 \mathbf{H}) = 0 \Rightarrow \mathbf{E}_e = -\mathbf{v} \times \mathbf{\mu}_0 \mathbf{H} \text{ inside}$$

- Open-circuit voltage across wire:
  $$\Phi = \mathbf{E}_e \cdot \mathbf{W} = \nu \mathbf{\mu}_0 \mathbf{H} \mathbf{W} [\mathbf{V}], \text{ where } \mathbf{W} = \text{wire length}$$

Electric fields inside conductors:

- Total force on free conduction electrons is produced by $\mathbf{v} \times \mathbf{H}$ plus an opposing $\mathbf{E}_e$
- $\mathbf{E}_e$ is produced by charge distributions that build inside wire until all electrons see $\mathbf{f} = 0$

MAGNETIC MOTORS ARE ALSO GENERATORS

Current $I$, force $f$, and power $P$ produced:

- $I = (V - \Phi) / R$
- $f = I \times \mu_0 \mathbf{H} \mathbf{W} = \nu \mu_0 \mathbf{H} \mathbf{W} (V - \Phi) / R$

Mechanical power delivered by wire from $V$:

$$P_m = f \cdot \mathbf{v} = \nu \mu_0 \mathbf{H} \mathbf{W} (V - \Phi) / R = \Phi (V - \Phi) / R \mathbf{W}$$

Electrical power delivered to wire by $V, R$:

$$P_e = VI - I^2 R = V (V - \Phi) / R - (V - \Phi)^2 / R$$

$$= [V (V - \Phi) / R] \mathbf{W} - (V - \Phi)^2 / R \mathbf{W}$$

Therefore: **Electrical power = Mechanical power**

It is motor if mechanical power out $> 0$: i.e. if $V > \Phi = \nu \mu_0 \mathbf{H} \mathbf{W}$

It is generator if electrical power out $> 0$: i.e. if $V < \Phi$, or $V > \nu \mu_0 \mathbf{H} \mathbf{W}$

i.e. when the motor “back voltage” $\Phi > V$.

In unloaded motors $V = \Phi$ and $I = 0$. 
**ROTARY WIRE MOTOR**

**Single wire loop spinning in uniform \( H \):**

- Total force (torque) is sum of forces on wire segments
- Axial forces from wires at ends cancel
- Tangential forces add \( \Rightarrow \) torque \( = 2f[m^1]W_r \)
  \( f = I\mu_0 H \) (\( = N\mu_0 H \) for \( N \)-turn coil)
  \( T = 2I\mu_0 H W_r = I\mu_0 H A \) [Nm], \( A \) is loop area
- Torque is a vector \( T = r \times f \)

**Torque varies with \( \theta \):**

\[
T[Nm] = I\mu_0 H(\theta)A
\]

If \( I = \) constant:
- With commutator

**Commutators:**
- Switch currents to maximize torque
- Can have \( N \) coils, \( 2N \) commutator segments

**TWO-POLE (N-S) COMMUTATED MOTOR**

**Design assumptions:**

\( \mu_0 H = \) constant \( = 1 \) Tesla (\( 10^4 \) gauss)
- One 100-turn loop (\( N = 100 \)) of area \( A = 10^{-3}[m^2] \)
- \( V = 24 \) volts, perfectly commutated

**Maximum \( \omega \), unloaded \( (T = 0) \):**

- \( \omega \) is angular frequency
- Back-voltage \( \Phi = V = 24[V] \) in equilibrium
  \( \Phi = 2NE_0 W = 2N\mu_0 H W = NA\mu_0 H_0 = 24 \)
  \( \omega = \frac{24}{NA\mu_0 H} = \frac{24}{(100 \times 10^{-3})} = 240 \Rightarrow 2292 \text{ rpm} \)
  (More typical values for \( B \) are 0.5 \( \Rightarrow \approx 4,600 \text{ rpm max} \))

**Maximum torque when \( \omega = 0 \):**

- Assume power supply is limited to \( I = 10[A] \)
  \( T = N\mu_0 H A = 100 \times 10 \times 1 \times 10^{-3} = 1 \) [Nm] \( \Rightarrow 100 \) N at \( r = 1 \text{ cm} \)
**Motor Torque/Power/Speed Relations**

Mechanical power output \( P = \omega T = \omega N I \mu_H \Pi A \)  
\( I = (V - \Phi) / R \)  
\( \Phi = N A \mu_H \omega \)  
\( P = \omega T = \omega N (V - N A \mu_H \omega) \mu_H / R \)

**Maximum mechanical power out \( P_p \):**

\( \delta P / \delta \omega = 0 \Rightarrow V = 2 \omega \gamma \mu_H \)  
[so at \( \omega_p \), we have \( \Phi = V / 2 \)]  
\( \omega_p = V / 2 N \mu_H \) [\( V = N A \mu_H \omega_{\text{max}} \)]  
\( P_p = \omega_p T = (V / 2 N \mu_H) N (V - [N A \mu_H V / 2 N \mu_H]) \mu_H / R = V^2 / 4R \)

At maximum power, motor is matched load of impedance \( R \)  
(\( \Phi = V / 2 \) is across motor, \( V / 2 \) is across \( R \))

Assume \( I_{\text{max}} = V / R = 192 \Omega \) and \( V = 24 \text{ volts} \). Then \( R = 0.125 \text{ ohms} \), and 
\( P_p = 24^2 / 4 \times 0.125 = 1.152 \text{ Kw} \)

**Motor design strategy:**

To minimize motor weight (cost), boost \( \omega N \mu_H \)

**Generator Power/Frequency Relations**

Electrical power out \( P = \omega T = \Phi I \):

\( \Phi = N A \mu_H \omega \)  
\( I = \Phi / R \)  
\( P = \Phi I = \Phi I / R = \omega^2 (N A \mu_H)^2 / R \)

To maximize power out \( \Rightarrow P_p \):

\( \omega \) is limited by vibration, lubricant, rotor fragmentation, and air viscosity (drag); smooth balanced rotors with \( \omega < c_b (~300 \text{ m/s}^{-1}) \) are best. \( \Phi < \) breakdown voltage. \( I \) limited by magnet survival (~200 C), insulation melting, and cooling; heat capacity allows transient peaks.

**Hall effect sensors:**

\( E_e = -v \times \mu_0 H \Rightarrow V_{\text{Hall}} = v \mu_0 H W \)

\( V_{\text{Hall}} \Rightarrow H \) if \( v \) is known; \( \Rightarrow v = l / nq \) if \( H \) is known (study carriers)