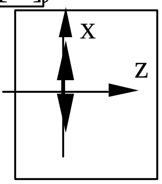
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

6.014 Electrodynamics

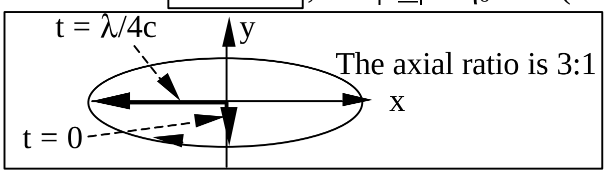
Problem Set 2 Solutions | Available in Tutorials: February 21, 2002

Problem 2.1

a) $\overline{\underline{E}} = 2\hat{x}e^{jy}$ propagates in the –y direction; $k = 2\pi/\lambda = 1$, so $\lambda = 2\pi$ [m]; $f = c/\lambda = 3\times10^8/2\pi =$ 47.7 MHz; $P = |\underline{\overline{E}}|^2/2\eta_o = 2^2/(2\times377) =$ 0.0053 W/m^2.



b)



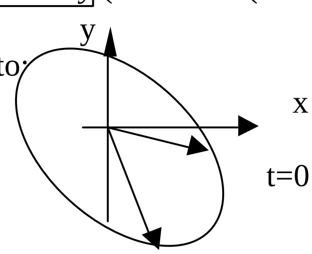
Problem 2.2

a) $\underline{a}(\hat{x} - j\hat{y}) + \underline{b}(\hat{x} + j\hat{y}) = 2\hat{x} + (1 - 5j)\hat{y}$

The x component $\Rightarrow R_e\{\underline{a} + \underline{b}) = 2$; $I_m\{\underline{a} + \underline{b}) = 0$; The y component $\Rightarrow R_e\{-\underline{a} + \underline{b}\} = -5$; and $I_m\{-\underline{a} + \underline{b}) = -1$. These four equations can be solved for the four unknowns, yielding $\underline{a} = (7 + j)/2$ and $\underline{b} = -(3 + j)/2$. This can obviously be generalized—any monochromatic wave can be similarly decomposed into two circularly polarized waves.

b) A quarter-wave delay is equivalent to multiplying by $e^{-j\pi/2} = -j$. So $\underline{b}$ becomes $\underline{c} = (-1 + 3j)/2$ and then $\underline{a}(\hat{x} - j\hat{y}) + \underline{c}(\hat{x} + j\hat{y}) = [(7+j)/2](\hat{x} - j\hat{y}) + [(-1 + 3j)/2](\hat{x} + j\hat{y}) = (3 + 2j)\hat{x} - (1 + 4j)\hat{y}$. The real part of the emerging $\overline{\underline{E}}$ then is:
$R_e\{\underline{E}\} = 3\hat{x} - \hat{y}$ (same as $\overline{E}(t=0)$), and $I_m\{\underline{E}\} = 2\hat{x} - 4\hat{y}$ (same as $\overline{E}(t = 3/4f)$).
This is right-elliptical polarization.

This corresponds to:



Note: $(3+2j)\hat{x} - (1+4j)\hat{y} \equiv \hat{x} - 1.144\angle 42.27^\circ\,\hat{y}$. Now Re{ $(\hat{x} - 1.144e^{j0.738}\hat{y})e^{jwt}$ } yields $\cos(wt)\hat{x} - \hat{y}1.144\sin(wt + 0.738)$ from which we can conclude that we have right- elliptical polarization (i.e., look at $wt = -.738$ and $wt = p/2$).

Problem 2.3

a) $|\overline{E}|^2/2\eta_o = 10^{12}$, so $\overline{E} = \hat{x}(2\eta_o 10^{12})^{0.5}\cos(\omega t - kz)$, where $\eta_o = 377\Omega$, $k = 2\pi/\lambda = 2\pi/10^{-6} = 6.28\times10^6$, and $\omega = 2\pi c/\lambda = 6\pi 10^{14} = 1.88\times10^{15}$ radians/s. Thus,
$\boxed{\overline{E}(z,t) = \hat{x}\,27.4\times10^6\cos(1.88\times10^{15}t - 6.28\times10^6 z)\ [\text{V/m}]}$, and
$\boxed{\overline{H}(z,t) = \hat{y}\,72.8\times10^3\cos(1.88\times10^{15}t - 6.28\times10^6 z)\ [\text{A/m}]}$ where $|\overline{E}|/|\overline{H}| = \eta_o$.

b) At $t = 0$ $\overline{E}$ is proportional to $2\cos(kz)$, so W_e is proportional to $2[\cos(2kz) + 1]$
$W_e(z) = \varepsilon_o|\overline{E}(z, t=0)|^2/2 = \boxed{6.65\times10^3[\cos(1.25\times10^7 z) + 1]\ [\text{J/m}^3]}$,
and $\overline{H}$ is proportional to $\cos(kz) - \cos(kz) = \boxed{0 = W_m(z)}$ at this instant of time.
At $z = 0$, $\overline{E}$ is proportional to $2\cos(\omega t)$; W_e is proportional to $2[\cos(2\omega t) + 1]$, so
$\boxed{W_e(t) = \varepsilon_o|E(z=0, t)|^2/2 = 6.65\times10^3[\cos(3.76\times10^{15}t) + 1]\ [\text{J/m}^3]}$
Here $\overline{H}$ is proportional to $\cos(\omega t) - \cos(\omega t) = \boxed{0 = W_m(t)}$ at this special position.

c) $\overline{S}(z=0, t) = \overline{E}\times\overline{H} = \hat{z}\,27.4\times10^6\times72.8\times10^3\cos^2(1.88\times10^{15}t)$
$= \hat{z}\,2\times10^{12}\cos^2(1.88\times10^{15}t) = \boxed{\hat{z}\,10^{12}[\cos(3.76\times10^{15}t) + 1]\ [\text{W/m}^2]}$
$\overline{S}(z, t=0) = \hat{z}\,10^{12}[\cos(1.25\times10^7 z) + 1]\ [\text{W/m}^2]$

d) $\underline{S} = \underline{E}\times\underline{H}^* = (\hat{x}\,27.4\times10^6)\times(\hat{y}\,72.8\times10^3)e^{-jkz+jkz} = \boxed{\hat{z}\,2\times10^{12}\ [\text{W/m}^2]}$
(Note that the time-average power flow is half the real part of $\underline{S}$.) For the standing wave, $\underline{E} = 27.4\times10^6(e^{-jkz} + e^{+jkz})$ and $\underline{H} = 72.8\times10^3(e^{-jkz} - e^{+jkz})$, so
$\underline{S} = \underline{E}\times\underline{H}^* = (\hat{x}\,27.4\times10^6)\times(\hat{y}\,72.8\times10^3)(e^{-jkz} + e^{+jkz})(e^{jkz} - e^{-jkz}) =$
$\hat{z}\,2\times10^{12}(e^{2jkz} - e^{-2jkz}) = \boxed{-\hat{z}\,j10^{12}\sin(26.28\times10^6 z)}$, which is purely imaginary, as we would expect for a pure standing wave.

Problem 2.4

a) Using the integral form of Gauss's law (see slide L4-2, or text p.48) for a spherically symmetric charge $\rho\,dV$ at the origin, we find $\overline{E}(r) = \hat{r}\,\rho\,dV/4\pi\varepsilon_o r^2 = \hat{r}\,Q/4\pi\varepsilon_o r^2$ where $r = 10^{-3}$ and $|\overline{E}| = 3\times10^7\ \text{vm}^{-1}$. Therefore $Q = 4\pi\varepsilon_o r^2\times3\times10^7 = 4\pi 8.85\times10^{-12}\times10^{-6}\times3\times10^7 = \boxed{\pm3.34\times10^{-9}\text{ coulombs.}}$ Note: to store a coulomb on a sphere would require a radius over 17 m. In contrast, a 1-kWh car battery delivers ~150,000 coulombs at 24 volts.

b) We can find the potential on the sphere by integrating the electric field $\overline{E}(r)$ from the sphere to infinity, where we define the potential as zero. $-\int_\infty^{0.001}(Q/4\pi\varepsilon_o r^2)dr = -(Q/4\pi\varepsilon_o)r^{-1}|_\infty^{0.001} = -10^3 Q/4\pi\varepsilon_o = \pm10^3\times3.34\times10^{-9}/4\pi 8.85\times10^{-12} \boxed{= \pm30\text{kV}}$.

Problem 2.5

a) Using (2.3.17) in the text, we see the total power radiated by a Hertzian dipole is $\eta_o|kId|^2/12\pi = 377|(2\pi/\lambda)^2\times 10^{-3}\times 0.1|^2/12\pi =$ $\boxed{4.39\times 10^{-11} \text{ watts,}}$ where $\lambda = c/f = 3\times 10^8/10^6 = 300$ meters.

b) The effective area of our matched receiving antenna $A = G\lambda^2/4\pi = 2\times 300^2/4\pi = 1.43\times 10^4$ m^2. Therefore our received power $P_{rec} = (G_tP_{rad}/4\pi r^2)A = 10^{-13}$ watts. Therefore, solving for r, we find

$$r = [(G_tP_{rad}/4\pi ME_b)A]^{0.5} = [(1.5\times 4.39\times 10^{11}/4\pi 10^{-13})1.43\times 10^4]^{0.5} = \boxed{866 \text{ m}}$$

We assumed we are radiating broadside, so the dipole gain G_t is 1.5.
($G_{dipole} = 1.5 \sin^2\theta$)