Problem 6.14 Electrodynamics

Problem 10.1

a) The definitions of the voltages and currents on the line as functions of position are: 
\[ V(z) = V_+ e^{-jkz} + V_- e^{jkz} \] and 
\[ I(z) = Y_0 (V_+ e^{-jkz} - V_- e^{jkz}) \] . At both ends of the line we have open ended terminations, hence reflection coefficients of one. Thus 
\[ I(z = 0) = 0 \Rightarrow V_+ = V_- \] and from this it follows that 
\[ V(z) = 2V_+ \cos(kz) \] and 
\[ I(z) = -j2Y_0 V_+ \sin(kz) \]. But we also need to have 
\[ k = \frac{n \pi c}{\ell} \in \mathbb{N} \] where \( c = \frac{3}{100} \) pF.

b) The energy stored in the resonator is twice the electric energy stored in the resonator. Thus 
\[ \omega_0 = \frac{\sqrt{LC}}{\ell} = 3 \times 10^8 \text{ } \forall n \in \{0, 1, \ldots\} \]
and we know that 
\[ c^{-1} = \sqrt{LC} \]. From (2) and (3) we obtain 
\[ C = \frac{100}{3} \text{ } \text{pF} \]
and substituting this into (1) we obtain 
\[ V_{\text{peak}} = 2V_+ = 346.4V \].

Place the resistor at \( z = \ell \) (could be placed at 0 as well). We are dealing with the case discussed in Section 8.5 in the paper, when a conductance is switched in at \( z = a = \ell \). We see that 
\[ G = 1 \times 10^{-4} \ll Y_0 = 1 \times 10^{-2} \] and we can thus assume that the voltage and current profiles are not significantly perturbed by adding the conductance. The conductance also does not change the resonant frequencies. The power dissipated in the resonator circuit is 
\[ P = \frac{1}{2} G |V(z = \ell)|^2 = 2G |V_+|^2 \cos^2(n\pi) \] and when we look at the first nonzero resonant frequency (\( n = 1 \)) we calculate 
\[ p = 2 \times 10^{-4} (173.2)^2 = 6 \text{ } \text{W} \]
from which we calculate 
\[ Q = \frac{\omega w_\tau}{P} = \frac{300\pi}{6} = 50\pi = 157.1 \]
Problem 10.2

a) Using the guidance condition, $k_z d = m\pi$ and the dispersion relation $k^2 = k_x^2 + k_z^2$, we can derive the cut-off frequency $\omega_m$, above which $k_z > 0$.

$$\omega_m = \frac{m\pi}{d \sqrt{\mu \varepsilon}}.$$

$TM_1 : f_{co} = \frac{\omega_1}{2\pi} = 0.75 \times 10^9 \text{ Hz}$

$TM_2 : f_{co} = \frac{\omega_2}{2\pi} = 1.5 \times 10^9 \text{ Hz}$

$TE_2 : f_{co} = \frac{\omega_2}{2\pi} = 1.5 \times 10^9 \text{ Hz}$

b) $\lambda_g = \frac{2\pi}{k_z} = \frac{2\pi}{\sqrt{\omega^2 \mu \varepsilon - (m\pi / d)^2}} = 1.15 \text{ cm}$

c) $v_g = \left( \frac{\partial k_z}{\partial \omega} \right)^{-1} = v \sqrt{1 - \left( \frac{m\pi v}{\omega d} \right)^2} = 1.3 \times 10^8 \text{ m/s}$