MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

# 6.014 Electrodynamics

Problem Set 12 Solutions    Available in Tutorials:    May 13, 2002

**Problem 12.1**

a) $$P = hf\frac{dn}{dt} = \frac{hfI}{e} = \frac{6.625\times10^{-34}\times3\times10^{14}\times10^{-3}}{1.6\times10^{-19}} = 1.24mW$$

b) $$\Delta f = \frac{v}{2L} = \frac{c}{6L} = \frac{3\times10^{8}\,m/s}{6\times10^{-3}\,m} = 50GHz$$

**Problem 12.2**

a) $$\frac{l_i}{l_o} = \frac{0.4mm}{1.6mm} = 0.25$$

b) $$n_2 = \infty$$
$$n_1 = n_3 = n_4 = 0$$

c) $A_{32}\ln 2$

**Problem 12.3**

a) $$\bar{f} = q\bar{E} = \frac{QE}{2} = \frac{CVE}{2} = \frac{eAV^2}{2d^2}$$ (Refer to L20-3 and Supplementary Lecture Notes 20 for further explanation on why the average electric field (E/2) is used.

b) $$\bar{f} = q_1\overline{E_1} + q_2\bar{E}_2 = \frac{Q_1E_1}{2} + \frac{Q_2E_2}{2} = \frac{C_1VE_1}{2} + \frac{C_2VE_2}{2} = \frac{eAV^2}{4d^2} + \frac{eAV^2}{16d^2} = \frac{5eAV^2}{16d^2}$$
where subscript 1 denotes the variables for plates with separation d Area = A/2 and subscript 2 denotes the variables for plates with separation 2d and Area = A/2.

c) $$f = -\frac{dE}{dx} = -\frac{1}{2}V^2\frac{d}{dx}(C_1 + C_2)$$
$$= -\frac{1}{2}V^2\left(\frac{eA}{2x} + \frac{eA}{2x}\right) = \frac{1}{2}V^2\left(\left.\frac{eA}{2x^2}\right|_{x=d} + \left.\frac{eA}{2x^2}\right|_{x=2d}\right) = \frac{5eAV^2}{16d^2}$$
Note: The negative sign is due to the fact that the energy stored in the system increases as the top plate moves in the –x direction.

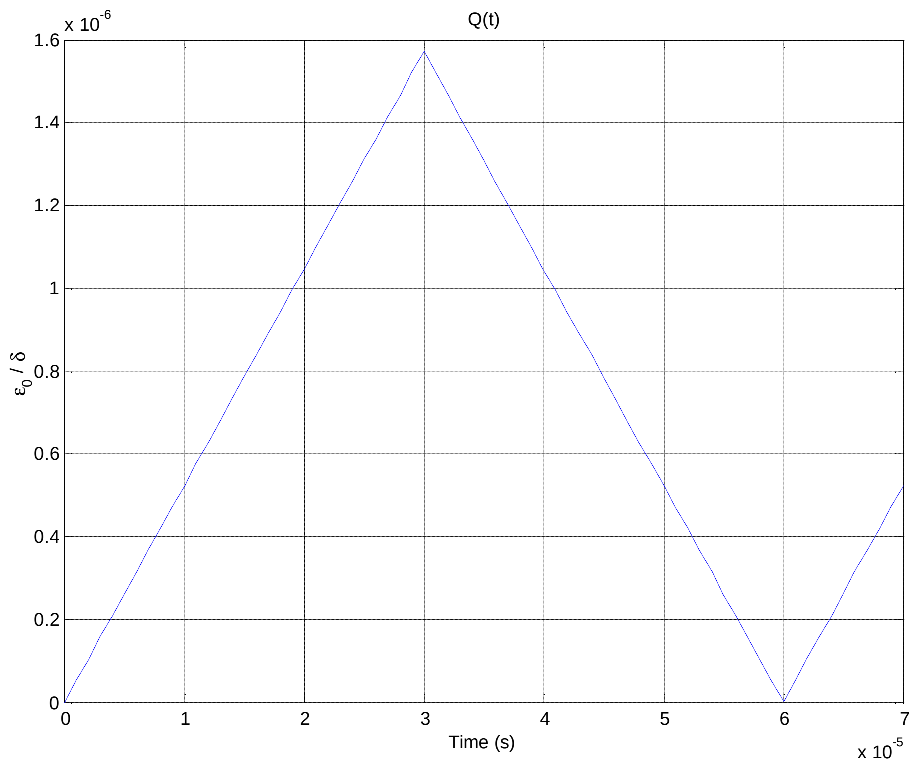
**Problem 12.4**

a) $Q(t) = C(t)V$, where $C(t)$ is an implicit function of time. $C(t) = \dfrac{\varepsilon_0 A(t)}{\delta}$ where

$$A(\theta) = \begin{cases} 2\times\left(\dfrac{1}{2}r^2\theta\right), \forall\theta\in[0,\pi/2]=\Omega_1 \\ 2\times\left(\dfrac{1}{2}r^2(\pi-\theta)\right), \forall\theta\in(\pi/2,\pi]=\Omega_2 \end{cases}$$

(where the 2x is due to two sectors that overlap, this is also the same as calculating the capacitance of one sector and then adding the two capacitors in parallel). The period for this area calculation is $\pi$. We also have that $\theta = \omega t + constant$ where the constant will be taken as zero in order to simplify things. Thus we have (where $\theta = \pi \rightarrow t = 60\mu s$)

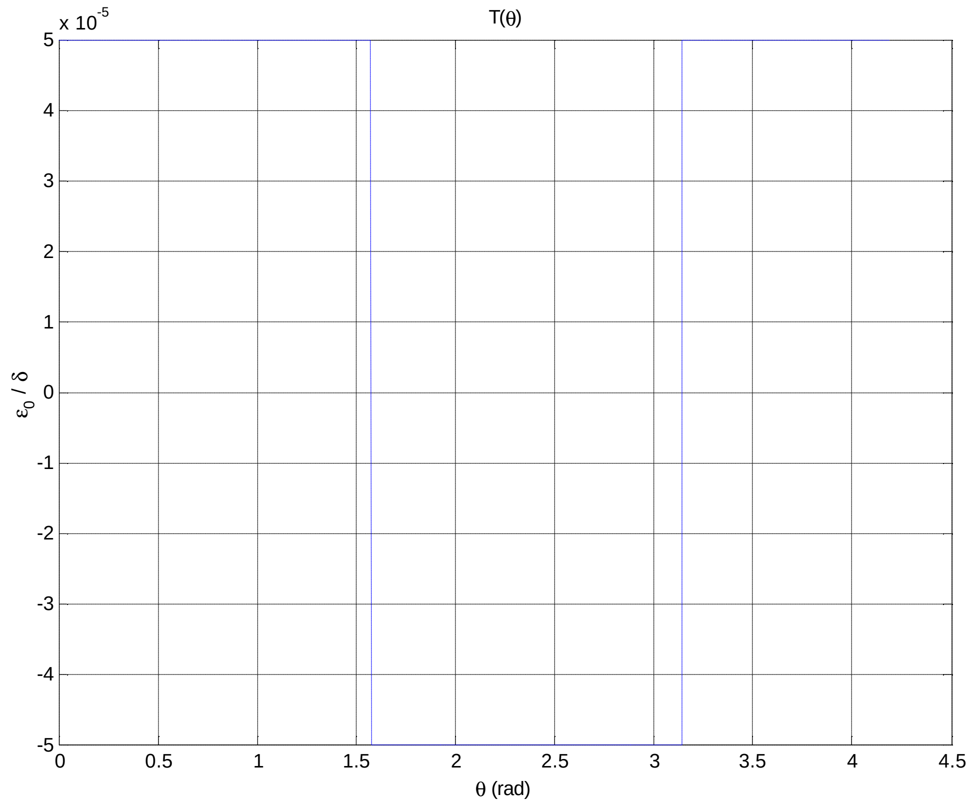
$$Q(t) = \begin{cases} 1\times10^{-5}\dfrac{\varepsilon_0}{\delta}\omega t, \forall t\in[0,30\mu s] \\ 1\times10^{-5}\dfrac{\varepsilon_0}{\delta}(\pi-\omega t), \forall t\in(30\mu s, 60\mu s] \end{cases}.$$




b) From L20-7 we have $T = -\dfrac{dw_T}{d\theta}$, where $dw_T$ is the increment by which the total system energy is increased. We will evaluate the above for the different regions of $\theta$. Thus we find that because of the changing area for the capacitor we get,

$$
dw_T = d(CV^2/2) - VdQ = -\frac{V^2 e_0}{2d}\, dA = \begin{cases} -\dfrac{V^2 r^2 e_0}{2d}\, dq, \forall q \in \Omega_1 \\ \dfrac{V^2 r^2 e_0}{2d}\, dq, \forall q \in \Omega_2 \end{cases}
$$

$$
\Rightarrow T(q) = \begin{cases} \dfrac{5\times 10^{-5} e_0}{d}, \forall q \in \Omega_1 \\ -\dfrac{5\times 10^{-5} e_0}{d}, \forall q \in \Omega_2 \end{cases}
$$




On average this machine is neither a motor nor a generator.

c) When the device operate as a motor the torque is positive. Thus the battery supply electrical power to the machine, which gets converted to mechanical power. When the torque is negative the machine is driven by a prime mover that insures that the shaft speed stays constant. For this instance the prime mover provides mechanical power to the machine and this machine converts the mechanical power to electrical power that gets delivered to the battery (that acts as a load to the electromechanical system). In the latter mode of operation the machine is classified as a generator. See the above sketch for the instances when the switch needs to flip from a to b and vice versa.

d) If we want to operate as a motor, the time averaged torque needs to stay positive. In order to maximize the time-averaged torque we want to switch to b when $\frac{\pi}{2} < \theta \le \pi$. Thus we need to extract the charge from the capacitor plates so that when $\pi < \theta \le \frac{3\pi}{2}$ we recharge the capacitor and $Q(t)$ increases from 0 to maximum charge as calculated in (a). There are two ways to go about finding a value for $R$:

- We need to make the time constant of the first order discharge RC circuit fast enough to extract all of the charge during $\frac{\pi}{2} < \theta \le \pi$. We know that after 5 time constants a first order response has settled to within 98% of its final value. So lets add a safety factor of 2, thus setting $\tau = RC_{\max} = 30\mu s / 5$ we can calculate a practical value for $R$. Note that the capacitance is a function of time and when $\theta = 0, \pi$ it is zero. It was decided to work with the maximum capacitance value. For this situation we have: $R = 6\frac{\delta}{\pi\varepsilon_0}$.
- We look at the RC circuit. The current is given as $i = \frac{dCV_c}{dt_b} = C\frac{dV_c}{dt_b} + V_c\frac{dC}{dt_b}$ where $V_c$ is the voltage across the plates of the capacitor for the time period of 30 microseconds to 60 microseconds; $t_b$ is a new time line and related to $t$ in such a fashion that when $t = 30\mu s \rightarrow t_b = 0; t = 60\mu s \rightarrow t_b = 30\mu s$. Using Kirchoff's Voltage law we can construct a ODE for the time evolution of the voltage across the capacitor plates. Once we obtain the solution of $V_c$ which is a function of time and $R$. By specifying how quickly (the quicker the better) we want $Q(t_b)$ to disappear we can solve for $R$.

e) Assuming that the negative torque portion of the cycle has been removed then we have that $\langle P_m \rangle = \frac{1}{60\times10^{-6}}\int_0^{60\times10^{-6}} T\omega dt \approx \frac{V^2 r^2 \varepsilon_0 \omega}{4\delta} = \frac{5\pi\varepsilon_0}{2\delta}$

f) The fraction of battery power converted to useful work when the switch is in position a is 100%. When the switch is in position b all the energy stored in the capacitor is to be dissipated by design. Seeing that the time period is the same (30 microseconds), the two powers are comparable.

g) When the switch is in position b we want to discharge the Capacitor into an Inductor (convert electric energy into magnetic energy). This magnetic energy is not dissipated as in the resistor case and this energy can be used to recharge the battery or help recharging the capacitor in the motor cycle by some switching device. We have a resonant tank with $\omega_0 = \frac{1}{\sqrt{LC}} \rightarrow T_0 = 2\pi\sqrt{LC}$ . Thus let us set $T_0 = 30\times10^{-6}$ and working again with the maximum capacitance of the device we can calculate an inductor value, $L = \left(\frac{30\times10^{-6}}{2\pi}\right)^2 \frac{1}{C_{\max}} = 4.5\times10^{-4}\ \frac{\delta}{\pi\varepsilon_0}$ .

**<u>Problem 12.5</u>**

a) We have a moving conductor in a static magnetic field. This problem also goes by the name, 'Faraday disk generator'. We start by looking at Lorentz's Force law. Noticing that we do not have an E-field, we get $\bar{f} = q(\bar{v}\times\bar{B})$. The magnetic force per unit charge is $(\bar{v}\times\bar{B})$, and can be interpreted as an induced electric field acting along the conductor (the disk( and producing a voltage. This voltage is called the flux cutting emf or motional emf. Thus $\boxed{V = \int_0^R (\bar{v}\times\bar{B}).d\hat{r} = \int_0^R (r\omega\hat{\theta}\times\mu_0 H\hat{z})\cdot d\hat{r} = \frac{\omega\mu_0 HR^2}{2}}$.

b) The battery is the load in this instance. There is a current $I$ flowing in the loop. The current density on the disk is $\bar{J}_s = -\frac{I}{2\pi dr}\hat{r}$ where $d$ is the thickness of the disk. The force density (given by equation (1.2.7) i.e., $\bar{F} = \rho_x\bar{E} + \bar{J}_s\times\bar{B}$ ) is calculated to be $\bar{F} = \hat{\theta}\frac{\mu_0 HI}{2\pi dr}$ . The force is tangential, thus in order to obtain the torque we multiply $\bar{F}$ by the lever arm $r$ and integrate over the volume of the disk. Hence $T = \int_0^d\int_0^{2\pi}\int_0^R (\frac{\mu_0 HI}{2\pi dr} r)\cdot rdr\cdot d\theta\cdot dz = \frac{\mu_0 HIR^2}{2}$ .

c) $P_m = T\omega = \frac{\omega\mu_0 HIR^2}{2}$

d) $P_e = VI = \frac{\omega\mu_0 HIR^2}{2} = P_m$. Thus there are no discrepancies, because there are no losses in this system.