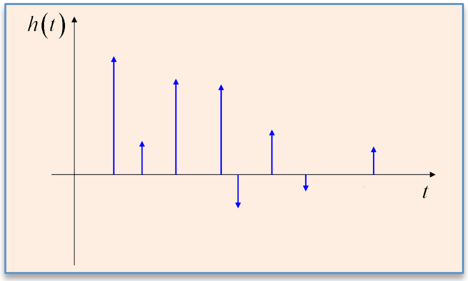
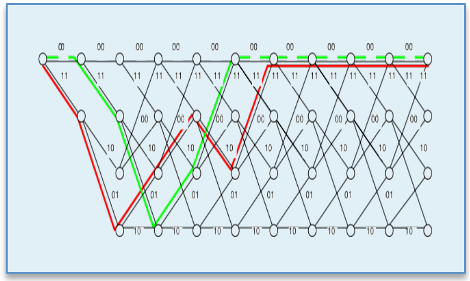
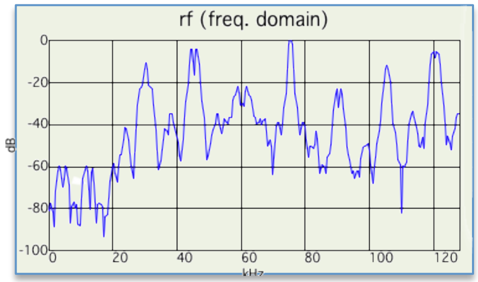
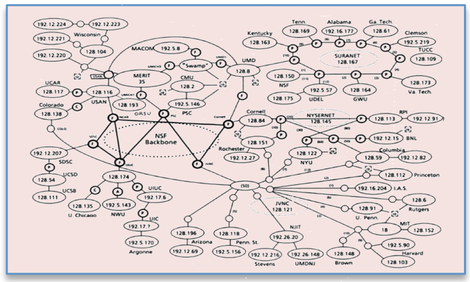








INTRODUCTION TO EECS II

DIGITAL COMMUNICATION SYSTEMS

6.02 Fall 2014
Lecture #9

- Bit detection in AWG noise
  - On-off vs. bipolar
  - Single sample versus average
  - Hard decision versus soft

Bit Detection in Noise

Recall that the receiver samples the value

$$y = x + w$$

in a particular bit slot (one sample per bit slot, for now),

where x is the transmitted value

$= V_0$ if the sender's codeword bit B=0 , probability P_0

$= V_1$ if the sender's codeword bit B=1 , probability P_1

$V_0 = 0$ and $V_1 = V$ for on-off signaling

$V_0 = -V$ and $V_1 = V$ for bipolar signaling

and

w is the value of the additive channel noise in this bit slot.

Conditional PDFs of received sample *Y*

- Think in terms of random variables,

$$Y = X + W$$

with each of these taking specific values *y, x, w* in the particular bit slot.

(*X* and *W* are assumed independent, i.e., knowing one tells us nothing about the other --- if the channel noise is signal-dependent, things get harder)

- How is *Y* distributed if *B=0 ?* → Described by **conditional PDF** $f_{Y|B}(y|0)$

- How is *Y* distributed if *B=1 ?* → Described by **conditional PDF** $f_{Y|B}(y|1)$

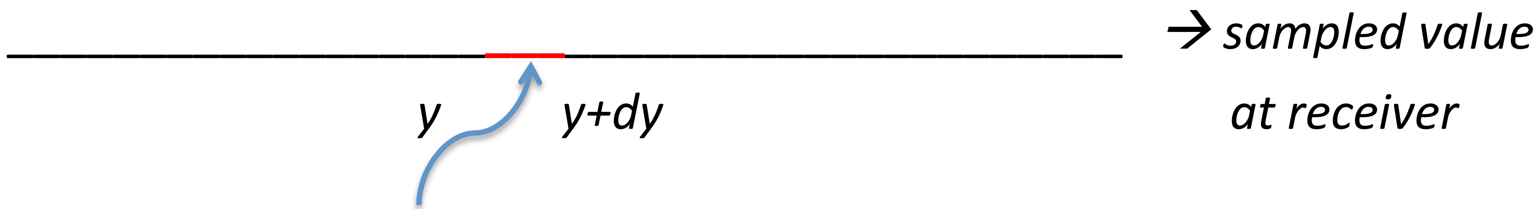
- What is $f_{Y|B}(y|0)$ if *W* is Gaussian, mean *0*, variance σ^2 ?

  Gaussian, mean V_0 , variance σ^2

- What is $f_{Y|B}(y|1)$ if *W* is Gaussian, mean *0*, variance σ^2 ?

  Gaussian, mean V_1 , variance σ^2

Bit Detection with Min Probability of Error



What is the probability that the received sample falls in this interval of length *dy* ?

$$f_{Y|B}(y|0)\,dy \quad \text{if} \quad B=0$$
$$f_{Y|B}(y|1)\,dy \quad \text{if} \quad B=1$$

What is the probability of error if receiver decides "0" when *y* lies here?

$$P_1 \,.\, f_{Y|B}(y|1)\,dy$$

What is the probability of error if receiver decides "1" when *y* lies here ?

$$P_0 \,.\, f_{Y|B}(y|0)\,dy$$

So, for min P(error) ···

- Decide "1" for all *y* where $P_1 \cdot f_{Y|B}(y|1) > P_0 \cdot f_{Y|B}(y|0)$
- Decide "0" for all *y* where $P_1 \cdot f_{Y|B}(y|1) < P_0 \cdot f_{Y|B}(y|0)$
- And the associated probability of error is

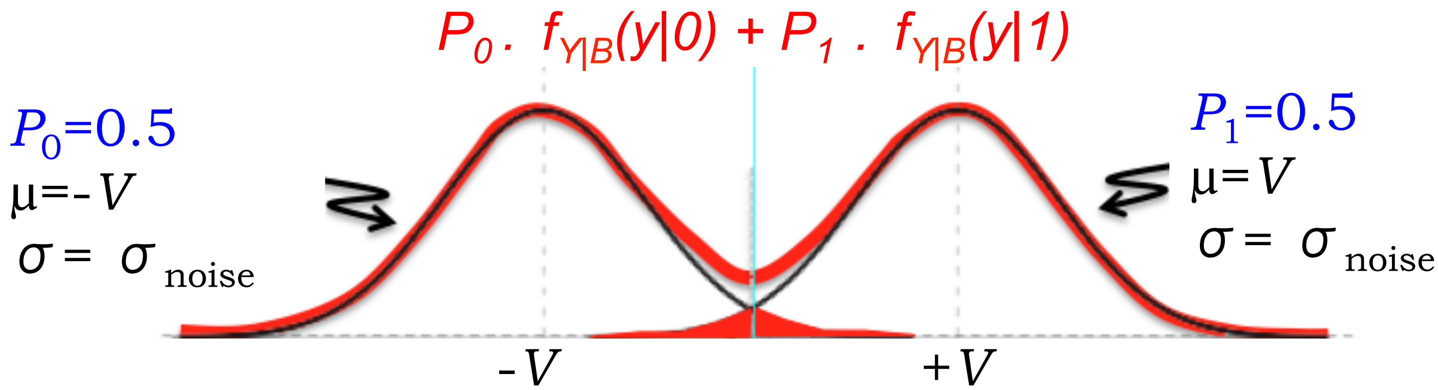
$\int P_1 \cdot f_{Y|B}(y|1)\, dy$ over the decision region for "0"

$+ \int P_0 \cdot f_{Y|B}(y|0)\, dy$ over the decision region for "1"

Simplifies when $P_1 = P_0$, and let's focus on that case.

- This is bit-by-bit detection, i.e., **hard detection** (not soft): the decision on this bit is made without regard for what's decided in other bit slots.

Connecting the "SNR" and BER for Bipolar Signaling



$$P(error) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_0^{\infty} e^{-(w-(-V))^2/(2\sigma^2)} dw$$

$$= Q\left(\frac{V}{\sigma}\right)$$

$$Q(1.0) = 0.159 \ , \quad Q(2.0) = 0.023 \ , \quad Q(3.0) = 0.001$$

How does this relate to *p* of BSC?

The Q(.) Function --- Area in the Tail of a Standard Gaussian

$$Q(t) = \frac{1}{\sqrt{2\pi}} \int_t^\infty e^{-v^2/2}\, dv$$

$$Q(-t) = 1 - Q(t)$$

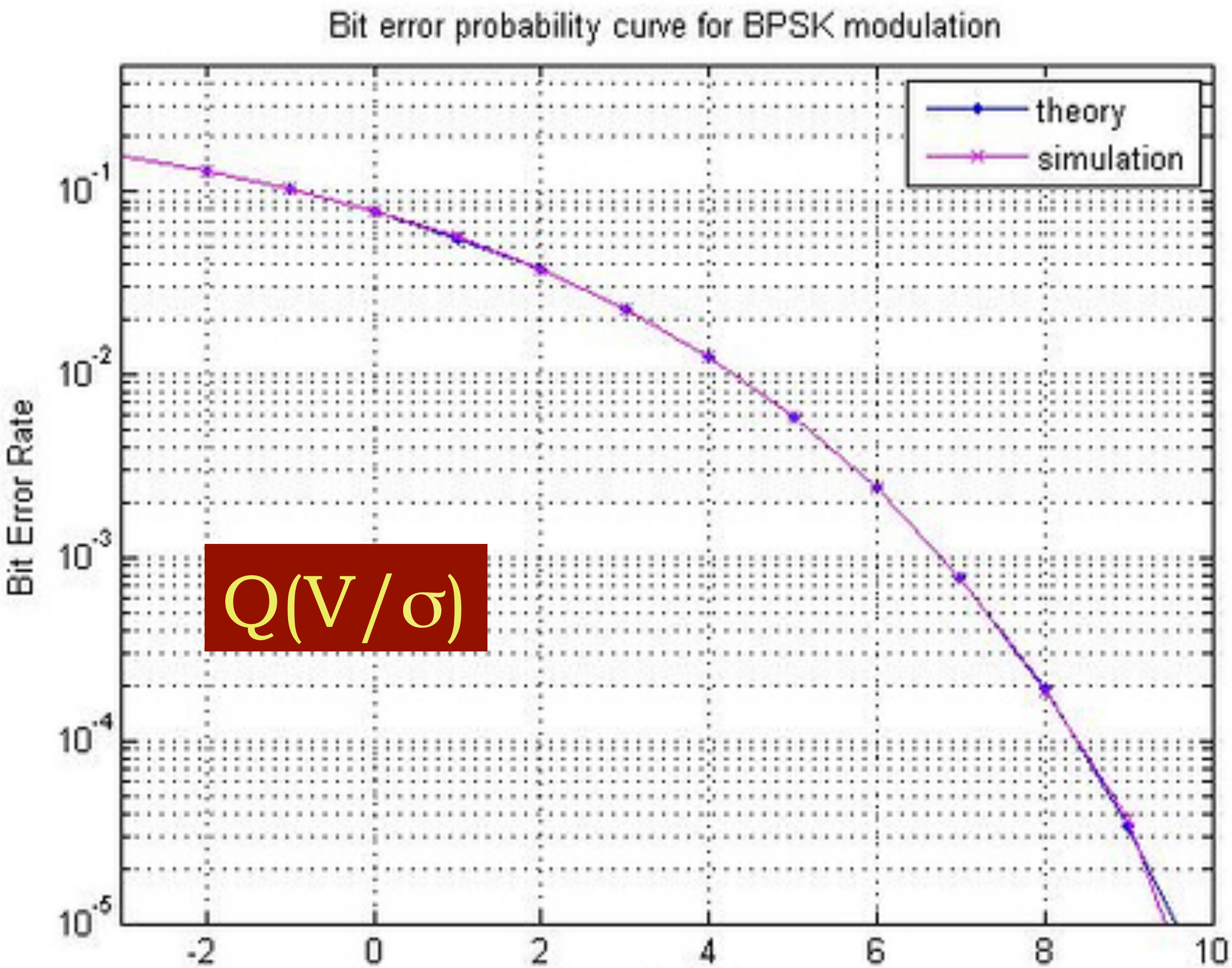
$$\frac{t}{(1+t^2)} \frac{e^{-t^2/2}}{\sqrt{2\pi}} < Q(t) < \frac{1}{t} \frac{e^{-t^2/2}}{\sqrt{2\pi}} \;, \qquad t > 0$$

Tail probability of a general Gaussian in terms of the Q(.) function

$$\frac{1}{\sqrt{2\pi\sigma^2}}\int_t^{\infty} e^{-(v-\mu)^2/(2\sigma^2)}\,dv$$

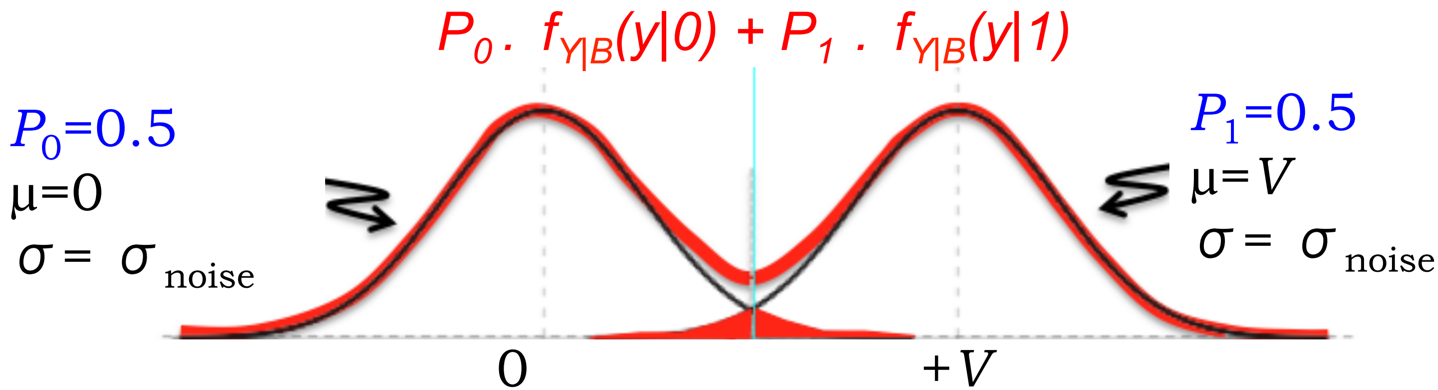
$$= Q\Big(\frac{t-\mu}{\sigma}\Big)$$

Bit Error Rate for Bipolar Signaling Scheme with Single-Sample Decision



$V^2/(2\sigma^2)$, dB

Comparison to On-Off Signaling



$$P(error) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{V/2}^{\infty} e^{-w^2/(2\sigma^2)} dw$$

$$= Q\left(\frac{V}{2\sigma}\right)$$

On-off has worse BER (e.g, if 0.023 for bipolar, then 0.159 for on-off), but half the average power (and average energy per bit), also simpler demodulation. Even increasing V to V√2 to use the same average power as bipolar, the on-off BER is worse, Q(V/(σ√2)).

Gaussian noise, $P_1 \neq P_0$

- Recall our optimal test:

- Decide "1" for all *y* where $P_1 \,.\, f_{Y|B}(y|1) \quad > \quad P_0 \,.\, f_{Y|B}(y|0)$
- Decide "0" for all *y* where $P_1 \,.\, f_{Y|B}(y|1) \quad < \quad P_0 \,.\, f_{Y|B}(y|0)$

So substitute in the expressions for the two Gaussian densities, simplify algebraically, and take the natural log of both sides (since the natural log is a monotone increasing function, this doesn't change the inequality signs). The result simplifies to:

Decide "1" if $y \;>\; \partial$ and decide "0" if $y \;<\; \partial$, where the threshold ∂

in the case of bipolar signaling is $\partial = (\sigma^2/(2V))\, ln(P_0/P_1)$

and for on-off signaling is $\partial = (V/2) + (\sigma^2/V)\, ln(P_0/P_1)$

(These expression simplify to the values we expect for $P_1 = P_0$)

Can we do better?

- Can we do better by amplifying the received measurement?

i.e., use

$$g.Y = g.(X+W)$$

to get

$$Y' = X' + W'$$

What are the two conditional densities involved now?

Have we improved things?

=> No, because SNR doesn't change: We multiply the means by g, but we also multiply the standard deviations of the noise by g.

But we can do better!

- Why just take a single sample from a bit slot?
- Instead, average M samples in the bit slot, with independent noise components:

  $y[n] = \pm V + w[n]$ *so*   **avg** $\{y[n]\} = \pm V +$ **avg** $\{w[n]\}$

(Recall that we are assuming no distortion, so the underlying signal values x[n] are assumed to be the same across all M samples.)

- **Claim: avg** {w[n]} is still Gaussian, still has mean 0, but its variance is now σ^2/M instead of σ^2 , so

  "SNR" is increased by a factor of M

(The sum of independent Gaussian random variables is Gaussian --- makes sense if you think of the central limit theorem representation of a Gaussian.)

- Same analysis and formulas as before, but now with $\sigma^2 \rightarrow \sigma^2/M$, or equivalently, sample energy $V^2 \rightarrow$ bit (or symbol) energy $E_b = M.V^2$ for same σ^2 as before. $Q(V/\sigma) \rightarrow Q(M^{1/2} V/\sigma)$

Implications for Signaling Rate

- As the noise intensity increases and/or signal strength decreases, we need to slow down the signaling rate, i.e., increase the number of samples per bit (M), to get a higher SNR in the samples extracted from a bit interval, if we wish to maintain the same error performance.
  - e.g. Voyager 2 was transmitting at 115 **kilobits**/s when it was near Jupiter in 1979. When it was over 9 billion miles away, 13 light hours away from the sun, twice as far away from the sun as Pluto, it was transmitting at only 160 **bits**/s. The received power at the Deep Space Network antennas on earth when Voyager was near Neptune was on the order of 10^(-16) watts!! --- 20 billion times smaller than an ordinary digital watch consumes. The received power now is estimated at less than 10^(-19) watts.

Flipped bits can have serious consequences!

- "On **November 30, 2006**, a telemetered command to *Voyager 2* was incorrectly decoded by its on-board computer—in a random error—as a command to turn on the electrical heaters of the spacecraft's magnetometer. These heaters remained turned on until December 4, 2006, and during that time, there was a resulting high temperature above 130 °C (266 °F), significantly higher than the magnetometers were designed to endure, and a sensor rotated away from the correct orientation. It has not been possible to fully diagnose and correct for the damage caused to the *Voyager 2's* magnetometer, although efforts to do so are proceeding."

- "On **April 22, 2010**, *Voyager 2* encountered scientific data format problems as reported by the Associated Press on May 6, 2010. On **May 17, 2010**, JPL engineers revealed that a flipped bit in an on-board computer had caused the issue, and scheduled a bit reset for May 19. On **May 23, 2010**, *Voyager 2* has resumed sending science data from deep space after engineers fixed the flipped bit."

http://en.wikipedia.org/wiki/Voyager_2

The moral of the story is ⋯

… if you're doing appropriate/optimal processing at the receiver, your effective SNR (and therefore your error performance) in the case of iid Gaussian noise is determined --- through the Q(.) function --- by the ratio of bit (or symbol) energy (not sample energy) to noise variance.

In the presence of channel distortion, optimum processing is more complicated than just averaging, because the received signal component r[n] no longer equals x[n], is no longer constant across the bit slot, and in fact contains contributions from bits in other slots (inter-symbol interference). This is dealt with optimally by more careful choice of x[n] and ("matched") filtering at the receiver before sampling – more than we have time for in 6.02. We shall proceed more pragmatically in this class: arrange signaling characteristics so that we have some number M of good samples for averaging in each bit slot.

AWGN Model for Noise Process *w[n]*

- Now let's look across multiple bits.

- Assume each *w[n]* is distributed as a Gaussian random variable *W*, with mean 0 and variance σ^2, and independently of *w[.]* at all other sample times

⇒ the iid Gaussian model, or

Additive White Gaussian Noise (AWGN) model

- For joint PDF, individual PDFs multiply for independent continuous random variables (just as probabilities multiply for independent discrete events) --- so in Gaussian case exponents add.

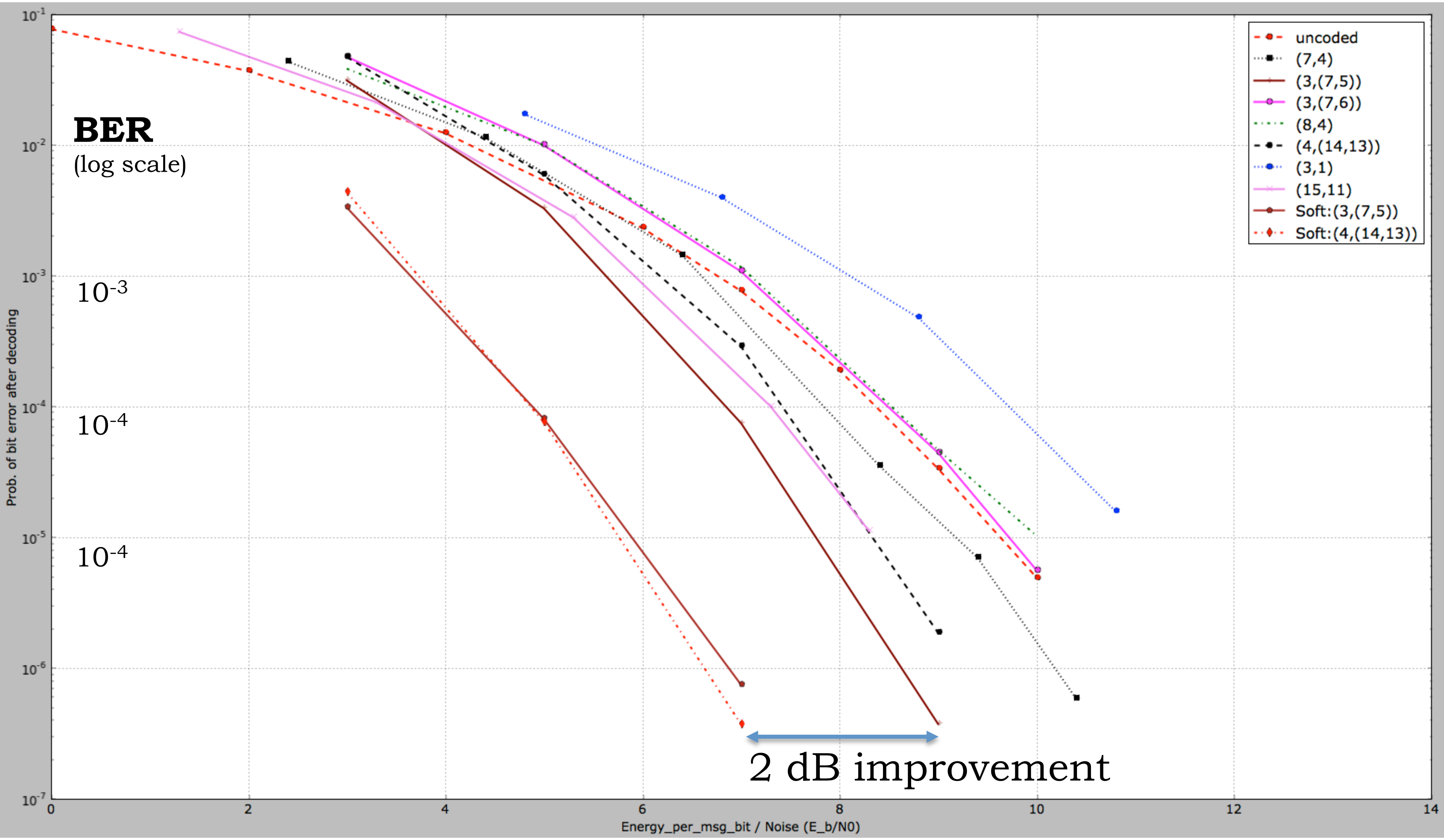
Soft-decision Decoding

We could defer decision at each bit slot, look at probability of whole sequence of bits, i.e., multiply probabilities, then maximize over all possible bit sequences => soft-decision decoding.

Or, equivalently, maximize log probability.

=> In the case of AWGN, this gives soft-decision decoding with sum-of-squares metric.

Soft Decoding Beats Hard Decoding



$V^2/(2\sigma^2)$, dB