6.02 Fall 2014
Lecture #12

• Frequency response
Transmission Over a Channel

$x[n]$ at 8 samples/bit

$y[n]$
Receiving the Response

Digitization threshold = 0.5V
Intersymbol Interference

$x[n]$ at 4 samples/bit

$y[n]$

Noise margin? $0.5 - y[28]$
Step response of audiocom channel
If system S is both linear and time-invariant (LTI), then we can use the unit sample response to predict the response to any input waveform x[n]:

Indeed, the unit sample response h[n] completely characterizes the LTI system S, so you often see

\[ x[n] \rightarrow h[.] \rightarrow y[n] \]
To Convolve (not “Convolute”!) 

\[ \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{m=-\infty}^{\infty} h[m]x[n-m] \]

A simple graphical implementation: 

Plot \( x[.] \) and \( h[.] \) as a function of the dummy index (\( k \) or \( m \) above) 

**Flip** (i.e., reverse) one signal in time, 
**slide** it right by \( n \) (slide left if \( n \) is –ve), take the 
**dot.product** with the other. 

This yields the value of the convolution at the single time \( n \). 

‘flip one & slide by \( n \) .... dot.product with the other’
Bounded-Input Bounded-Output (BIBO) Stability

What ensures that the infinite sum

$$y[n] = \sum_{m=-\infty}^{\infty} h[m] x[n-m]$$

is well-behaved?

One important case: If the unit sample response is absolutely summable, i.e.,

$$\sum_{m=-\infty}^{\infty} |h[m]| < \infty$$

and the input is bounded, i.e.,

$$|x[k]| \leq M < \infty$$

Under these conditions, the convolution sum is well-behaved, and the output is guaranteed to be bounded.

The absolute summability of $h[n]$ is necessary and sufficient for this bounded-input bounded-output (BIBO) stability.
Time Now for a Frequency-Domain Story

in which convolution is transformed to multiplication, and other good things happen
A First Step

Do periodic inputs to an LTI system, i.e., \( x[n] \) such that

\[
x[n+P] = x[n] \text{ for all } n, \text{ some fixed integer } P > 0
\]

(with \( P \) usually picked to be the smallest positive integer for which this is true) yield periodic outputs? If so, of period \( P \)?

Yes! --- Since the system is TI, using input \( x \) delayed by \( P \) should yield \( y \) delayed by \( P \). But \( x \) delayed by \( P \) is \( x \) again, so \( y \) delayed by \( P \) must be \( y \). (Linearity is not needed.)

Alternate argument: use Flip/Slide/Dot.Product to see this easily: sliding by \( P \) gives the same picture back again, hence the same output value.
But much more is true for **Sinusoidal Inputs to LTI Systems**

Sinusoidal inputs, i.e.,

\[ x[n] = \cos(\Omega n + \phi) \]

yield sinusoidal outputs at the same angular ‘frequency’ \( \Omega \) rads/sample.

Note that such inputs are not even periodic in general.

Periodic if and only if \( 2\pi/\Omega \) is rational, \( =P/Q \) for some integers \( P(>0), Q \). The smallest such \( P \) is the period.

Nevertheless, we often refer to \( 2\pi/\Omega \) as the ‘period’ of this sinusoid, whether or not it is a periodic discrete-time sequence. This is the period of an underlying continuous-time sinusoid that, sampled at integer times, produces the given discrete-time sequence.
Examples

\[ \cos\left(\frac{3\pi n}{4}\right) \text{ has frequency } \frac{3\pi}{4} \text{ rad/sample, and} \]
\[ \text{period } 8; \text{ shifting by integer multiples of } 8 \text{ yields the same} \]
\[ \text{sequence back again, and no integer smaller than} \]
\[ 8 \text{ accomplishes this.} \]

\[ \cos\left(\frac{3n}{4}\right) \text{ has frequency } \frac{3}{4} \text{ rad/sample, and is not periodic as} \]
\[ \text{a DT sequence because } 8\pi/3 \text{ is irrational, but we could} \]
\[ \text{still refer to } 8\pi/3 \text{ as its ‘period’, because we can} \]
\[ \text{think of the sequence as arising from sampling the} \]
\[ \text{periodic continuous-time signal } \cos\left(\frac{3t}{4}\right) \text{ at integer } t. \]
A very important property of LTI systems or channels:

If the input $x[n]$ is a sinusoid of a given amplitude, frequency and phase, the response will be a **sinusoid at the same frequency**, although the amplitude and phase may be altered. The change in amplitude and phase will, in general, depend on the frequency of the input.

Let’s prove this to be true ... but use **complex exponentials** instead, for clean derivations that take care of sines and cosines (or sinusoids of arbitrary phase) simultaneously.
A related simple case: real discrete-time (DT) exponential inputs also produce exponential outputs of the same type

- Suppose \( x[n] = r^n \) for some real number \( r \)

\[
y[n] = \sum_{m=-\infty}^{\infty} h[m] x[n-m]
= \sum_{m=-\infty}^{\infty} h[m] r^{n-m}
= \left( \sum_{m=-\infty}^{\infty} h[m] r^{-m} \right) r^n
\]

- i.e., just a scaled version of the exponential input
Euler’s formula shows the relation between complex exponentials and our usual trig functions:

\[ e^{j\varphi} = \cos(\varphi) + j\sin(\varphi) \]

\[ \cos(\varphi) = \frac{1}{2} e^{j\varphi} + \frac{1}{2} e^{-j\varphi} \quad \text{and} \quad \sin(\varphi) = \frac{1}{2j} e^{j\varphi} - \frac{1}{2j} e^{-j\varphi} \]

In the complex plane, \( e^{j\varphi} = \cos(\varphi) + j\sin(\varphi) \) is a point on the unit circle, at an angle of \( \varphi \) with respect to the positive real axis. cos and sin are projections on real and imaginary axes, respectively.

Increasing \( \varphi \) by \( 2\pi \) brings you back to the same point!
So any function of \( e^{j\varphi} \) only needs to be studied for \( \varphi \) in \([-\pi, \pi]\).
Useful Properties of $e^{j\varphi}$

When $\varphi = 0$:
$$e^{j0} = 1$$

When $\varphi = \pm \pi$:
$$e^{j\pi} = e^{-j\pi} = -1$$
$$e^{j\pi n} = e^{-j\pi n} = (-1)^n$$

(More properties later)
Frequency Response

\[
A(\cos \Omega n + j \sin \Omega n) = A e^{j\Omega n} \rightarrow \begin{array}{c} h[.] \\ \rightarrow y[n] \end{array}
\]

Using the convolution sum we can compute the system’s response to a complex exponential (of frequency \(\Omega\)) as input:

\[
y[n] = \sum_m h[m]x[n-m]
\]

\[
= \sum_m h[m] Ae^{j\Omega(n-m)}
\]

\[
= \left( \sum_m h[m] e^{-j\Omega m} \right) Ae^{j\Omega n}
\]

\[
= H(\Omega) \cdot x[n]
\]

where we’ve defined the \textit{frequency response} of the system as

\[
H(\Omega) \equiv \sum_m h[m] e^{-j\Omega m}
\]
From Complex Exponentials to Sinusoids

\[
\cos(\Omega n) = \frac{(e^{j\Omega n} + e^{-j\Omega n})}{2}
\]

So response to this cosine input is

\[
\frac{(H(\Omega)e^{j\Omega n} + H(-\Omega)e^{-j\Omega n})}{2} = \text{Real part of } H(\Omega)e^{j\Omega n}
\]

\[
= \text{Real part of } |H(\Omega)|e^{j(\Omega n + <H(\Omega))}
\]

This is IMPORTANT
Example $h[n]$ and $H(\Omega)$

Sometimes written as $H(e^{j\Omega})$
Frequency Response of “Moving Average” Filters
Relating Time- and Frequency-Domain Properties

- Why does the 5-point moving average filter on the preceding page have nulls in its frequency response magnitude at ±0.4π and ±0.8π?

(Think of convolving the unit sample response of this filter with sinusoids at these frequencies.)
Convolution in Time $\iff$ Multiplication in Frequency

In the frequency domain (i.e., thinking about input-to-output frequency response):

\[ H(\Omega) = H_2(\Omega)H_1(\Omega) \]

i.e., convolution in time has become multiplication in frequency!
Example: “Deconvolving” Output of Channel with Echo

Suppose channel is LTI with

\[ h_1[n] = \delta[n] + 0.8\delta[n-1] \]

\[ H_1(\Omega) = ?? = \sum_{m} h_1[m]e^{-j\Omega m} \]

\[ = 1 + 0.8e^{-j\Omega} = 1 + 0.8\cos(\Omega) - j0.8\sin(\Omega) \]

So:

\[ |H_1(\Omega)| = [1.64 + 1.6\cos(\Omega)]^{1/2} \quad \text{EVEN function of } \Omega; \]

\[ <H_1(\Omega) = \arctan \left( -\frac{0.8\sin(\Omega)}{1 + 0.8\cos(\Omega)} \right) \quad \text{ODD}. \]
Given $H_1(\Omega)$, what should $H_2(\Omega)$ be, to get $z[n]=x[n]$?

$$H_2(\Omega) = \frac{1}{H_1(\Omega)}$$  “Inverse filter”

$$= \left(\frac{1}{|H_1(\Omega)|}\right) \cdot \exp\{-j\angle H_1(\Omega)\}$$

Inverse filter at receiver can do very badly in the presence of noise that adds to $y[n]$:

filter has high gain for noise precisely at frequencies where channel gain $|H_1(\Omega)|$ is low (and channel output is weak)!
A Deeper Reason for Interest in Sinusoidal Inputs

- General inputs $x[.]$ can be written as “sums” of sinusoids

- Each input sinusoidal component is mapped via the frequency response $H(\Omega)$ to its corresponding sinusoidal output component

- Superposition of these output components yields the general response $y[.]$

We’ll develop this story over the next couple of lectures.