6.02 Fall 2014 Lecture #13

- Frequency response
- Filters
- Introduction to spectral content of signals
A very important property of LTI systems or channels:

If the input \( x[n] \) is a sinusoid of a given amplitude, frequency and phase, the response will be a *sinusoid at the same frequency*, although the amplitude and phase may be altered. The change in amplitude and phase will, in general, depend on the frequency of the input.
Complex Exponentials as “Eigenfunctions” of LTI System

\[ x[n] = e^{j\Omega n} \quad \rightarrow \quad h[.] \quad \rightarrow \quad y[n] = H(\Omega) e^{j\Omega n} \]

Eigenfunction: Undergoes only scaling -- by the frequency response \( H(\Omega) \) in this case:

\[
H(\Omega) \equiv \sum_{m} h[m] e^{-j\Omega m} \\
= \sum_{m} h[m] \cos(\Omega m) - j \sum_{m} h[m] \sin(\Omega m)
\]

This is an infinite sum in general, but is well behaved if \( h[.] \) is absolutely summable, i.e., if the system is stable.

We also call \( H(\Omega) \) the **discrete-time Fourier transform (DTFT)** of the time-domain function \( h[.] \) --- more on the DTFT later.
From Complex Exponentials to Sinusoids

\[
\cos(\Omega n) = (e^{j\Omega n} + e^{-j\Omega n}) / 2
\]

So response to a cosine input is:

\[
\text{Acos}(\Omega_0 n + \phi_0) \rightarrow |H(\Omega)| \text{Acos}(\Omega_0 n + \phi_0 + <H(\Omega_0))
\]

(Recall that we only need vary \( \Omega \) in the interval \([-\pi, \pi]\).)

This gives rise to an easy experimental way to determine the frequency response of an LTI system.
Loudspeaker Frequency Response

SPL Versus Frequency
(Speaker Sensitivity = 85dB)

-3dB @ 56.5 Hz
-3dB @ 12.5kHz
Connection between CT and DT

The continuous-time (CT) signal

\[ x(t) = \cos(\omega t) = \cos(2\pi f t) \]

sampled every \( T \) seconds, i.e., at a sampling frequency of \( f_s = 1/T \), gives rise to the discrete-time (DT) signal

\[ x[n] = x(nT) = \cos(\omega nT) = \cos(\Omega n) \]

So

\[ \Omega = \omega T \]

and \( \Omega = \pi \) corresponds to \( \omega = \pi/T \) or \( f = 1/(2T) = f_s/2 \)
Properties of $H(\Omega)$

Repeats periodically on the frequency (\(\Omega\)) axis, with period $2\pi$, because the input $e^{j\Omega n}$ is the same for $\Omega$ that differ by integer multiples of $2\pi$. So only the interval $\Omega$ in $[-\pi, \pi]$ is of interest!
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$\Omega = 0$, i.e., $e^{j\Omega n} = 1$, corresponds to a constant (or “DC”, which stands for “direct current”, but now just means constant) input, so $H(0)$ is the “DC gain” of the system, i.e., gain for constant inputs.

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$\Omega = \pi$ or $-\pi$, i.e., $Ae^{j\Omega n} = (-1)^n A$, corresponds to the highest-frequency variation possible for a discrete-time signal, so $H(\pi) = H(-\pi)$ is the high-frequency gain of the system.

$$H(\pi) = \sum (-1)^m h[m] \quad \text{--- show from definition!}$$
Symmetry Properties of $H(\Omega)$

\[ H(\Omega) \equiv \sum_{m} h[m] e^{-j\Omega m} \]
\[ = \sum_{m} h[m] \cos(\Omega m) - j \sum_{m} h[m] \sin(\Omega m) \]
\[ = C(\Omega) - jS(\Omega) \]

For real $h[n]$:
- **Real part** of $H(\Omega)$ & **magnitude** are EVEN functions of $\Omega$.
- **Imaginary part** & **phase** are ODD functions of $\Omega$.

For real and **even** $h[n] = h[-n]$, $H(\Omega)$ is purely real.
For real and **odd** $h[n] = -h[-n]$, $H(\Omega)$ is purely imaginary.
Convolution in Time $\Longleftrightarrow$ Multiplication in Frequency

In the frequency domain (i.e., thinking about input-to-output frequency response):

\[ H(\Omega) = H_2(\Omega)H_1(\Omega) \]

i.e., convolution in time has become multiplication in frequency!
Example: “Deconvolving” Output of Channel with Echo

Suppose channel is LTI with

\[ h_1[n] = \delta[n] + 0.8\delta[n-1] \]

\[
H_1(\Omega) = \sum_{m} h_1[m] e^{-j\Omega m} = 1 + 0.8e^{-j\Omega} = 1 + 0.8\cos(\Omega) - j0.8\sin(\Omega)
\]

So:

\[ |H_1(\Omega)| = [1.64 + 1.6\cos(\Omega)]^{1/2} \quad \text{EVEN function of } \Omega; \]

\[ <H_1(\Omega) = \arctan \left[-(0.8\sin(\Omega))/[1 + 0.8\cos(\Omega)]\right] \quad \text{ODD}. \]
A Frequency-Domain view of Deconvolution

Given $H_1(\Omega)$, what should $H_2(\Omega)$ be, to get $z[n] = x[n]$?

$H_2(\Omega) = \frac{1}{H_1(\Omega)}$  

“Inverse filter”

$= \frac{1}{|H_1(\Omega)|} \cdot \exp(-jH_1(\Omega))$

Inverse filter at receiver typically does very badly in the presence of noise that adds to $y[n]$:  
filter has high gain for noise precisely at frequencies where channel gain $|H_1(\Omega)|$ is low (and channel output is weak)!
Illustrative Frequency Responses of Channel Models

- **h[n] for fast channel**
  - Frequency response of a fast channel.

- **h[n] for slow channel**
  - Frequency response of a slow channel.

- **h[n] for ringing channel**
  - Frequency response of a ringing channel.
To Get a Filter with a Specified Zero-Pair in $H(\Omega)$

- Let $h[0] = h[2] = 1, \ h[1] = \mu, \ \text{all other } h[n] = 0$

- Then $H(\Omega) = 1 + \mu e^{-j\Omega} + e^{-j2\Omega} = e^{-j\Omega} (\mu + 2\cos(\Omega))$

- So $|H(\Omega)| = |\mu + 2\cos(\Omega)|$, with zeros at $\pm \arccos(-\mu/2)$
A 10-cent Lowpass Filter

Suppose we wanted a lowpass filter with a cutoff frequency of $\pi/4$?

$x[n] \rightarrow H_{\pi/4}(\Omega) \rightarrow H_{\pi/2}(\Omega) \rightarrow H_{3\pi/4}(\Omega) \rightarrow H_{\pi}(\Omega) \rightarrow y[n]$
The $4.99 version of a Lowpass Filter, $h[n]$ and $H(\Omega)$
Determining $h[n]$ from $H(\Omega)$

$$H(\Omega) = \sum_m h[m] e^{-j\Omega m}$$

Multiply both sides by $e^{j\Omega n}$ and integrate over a (contiguous) $2\pi$ interval. Only one term survives!

$$\int_{<2\pi>} H(\Omega) e^{j\Omega n} d\Omega = \int_{<2\pi>} \sum_m h[m] e^{-j\Omega (m-n)} d\Omega$$

$$= 2\pi \cdot h[n]$$

$$h[n] = \frac{1}{2\pi} \int_{<2\pi>} H(\Omega) e^{j\Omega n} d\Omega$$
Design ideal lowpass filter with cutoff frequency $\Omega_c$ and $H(\Omega) = 1$ in passband

\[
h[n] = \frac{1}{2\pi} \int_{-2\pi}^{2\pi} H(\Omega) e^{j\Omega n} d\Omega
\]

\[= \frac{1}{2\pi} \int_{-\Omega_c}^{\Omega_c} 1 \cdot e^{j\Omega n} d\Omega
\]

\[= \frac{\sin(\Omega_c n)}{\pi n}, \quad n \neq 0
\]

\[= \Omega_c / \pi, \quad n = 0
\]

DT “sinc” function (extends to $\pm\infty$ in time, falls off only as $1/n$)
Exercise: Frequency response of $h[n-D]$ 

Given an LTI system with unit sample response $h[n]$ and associated frequency response $H(\Omega)$, 

determine the frequency response $H_D(\Omega)$ of an LTI system whose unit sample response is 

$$h_D[n] = h[n-D].$$

**Answer:** 

$$H_D(\Omega) = \exp\{-j\Omega D\}.H(\Omega)$$

so : 

$$|H_D(\Omega)| = |H(\Omega)|$$, i.e., magnitude unchanged 

$$<H_D(\Omega) = -\Omega D + <H(\Omega)$$, i.e., linear phase term added
e.g.: Approximating an ideal lowpass filter

Idea: shift $h[n]$ right to get causal LTI system. Will the result still be a lowpass filter?
Causal approximation to ideal lowpass filter

\[ h_C[n] = h[n-300] \]

Determine \(<H_C(\Omega)\)
Useful Filters
A Deeper Reason for Interest in Sinusoidal Inputs

• General inputs $x[.]$ can be written as “sums” of sinusoids

• Each input sinusoidal component is mapped via the frequency response $H(\Omega)$ to its corresponding sinusoidal output component

• Superposition of these output components yields the general response $y[.]$

We’ll develop this story over the next couple of lectures, but here’s a start →