• Filtering
• Spectral content of signals via the DTFT
Single Link Communication Model

Original source

Digitize (if needed)

Source coding

Source symbols

Bit stream

End-host devices

Receiving app/user

Render/display, etc.

Source decoding

Bit stream

Channel coding (for bit error protection)

Mapper + Xmit samples

Recv samples + Demapper

Channel decoding (bit error correction)

SIGNALS!
Modeling The Baseband Channel

codeword bits in
1001110101

generate digitized symbols

DAC

modulate

ADC

demodulate & filter

sample & threshold

1001110101
codeword bits out

NOISY & DISTORTING ANALOG CHANNEL
The LTI Baseband** Channel Model

Frequency response \( H(\Omega) \) describes the response to sinusoids (or equivalently, to complex exponentials, i.e., “cisoids”)

Similar math for LTI filtering of \( y[n] \) for downstream processing

**From before the modulator till after the demodulator & filter
A Deeper Reason for Interest in Sinusoidal Inputs

• General inputs $x[.]$ can be written as linear combinations of sinusoids (or complex exponentials)

• Each input sinusoidal component is mapped via the frequency response $H(\Omega)$ to its corresponding sinusoidal output component

• Superposition of these output components yields the general response $y[.]$

We’ll see this in today’s lecture.
Determining $h[n]$ from $H(\Omega)$

$$H(\Omega) = \sum_{m} h[m] e^{-j\Omega m}$$

Multiply both sides by $e^{j\Omega n}$ and integrate over a (contiguous) $2\pi$ interval. Only one term survives!

$$\int_{<2\pi>} H(\Omega)e^{j\Omega n} \, d\Omega = \int_{<2\pi>} \sum_{m} h[m] e^{-j\Omega (m-n)} \, d\Omega$$

$$= 2\pi \cdot h[n]$$

$$h[n] = \frac{1}{2\pi} \int_{<2\pi>} H(\Omega)e^{j\Omega n} \, d\Omega$$
Design ideal lowpass filter with cutoff frequency $\Omega_c$ and $H(\Omega) = 1$ in passband

$$h[n] = \frac{1}{2\pi} \int_{-2\pi}^{2\pi} H(\Omega) e^{j\Omega n} d\Omega$$

$$= \frac{1}{2\pi} \int_{-\Omega_c}^{\Omega_c} 1 \cdot e^{j\Omega n} d\Omega$$

$$= \frac{\sin(\Omega_c n)}{\pi n}, \; n \neq 0$$

$$= \frac{\Omega_c}{\pi}, \; n = 0$$

DT “sinc” function
(extends to $\pm \infty$ in time, falls off only as $1/n$)
Approximating an ideal lowpass filter

Idea: shift $h[n]$ right to get causal LTI system.
Will the result still be a lowpass filter?
Causal approximation to ideal lowpass filter

$$h_C[n] = h[n-300]$$

$$|H_C[\Omega]|$$

Determine $$<H_C(\Omega)$$
Lowpass filtering (10 Hz cutoff) of blood flow velocity in middle cerebral artery, measured using transcranial Doppler ultrasound
Useful Filters
DT Fourier Transform (DTFT) for Spectral Representation of General \( x[n] \)

If we can write

\[
h[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(\Omega) e^{j\Omega n} d\Omega \quad \text{where} \quad H(\Omega) = \sum_{m} h[n] e^{-j\Omega m}
\]

then we can write

\[
x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega) e^{j\Omega n} d\Omega \quad \text{where} \quad X(\Omega) = \sum_{m} x[n] e^{-j\Omega m}
\]

Any contiguous interval of length \( 2\pi \)

This Fourier representation expresses \( x[n] \) as a weighted combination of \( e^{j\Omega n} \) for all \( \Omega \) in \([-\pi, \pi]\).

\( X(\Omega_o) d\Omega \) is the spectral content of \( x[n] \) in the frequency interval \([\Omega_o, \Omega_o + d\Omega]\).
The spectrum of the exponential signal \((0.5)^nu[n]\) is shown over the frequency range \(\Omega = 2\pi f\) in \([-4\pi, 4\pi]\), The angle has units of degrees.
Fast Fourier Transform (FFT) to compute samples of the DTFT for signals of finite duration

\[ X(\Omega_k) = \sum_{m=0}^{P-1} x[m] e^{-j\Omega_km}, \quad x[n] = \frac{1}{P} \sum_{k=-P/2}^{(P/2)-1} X(\Omega_k) e^{j\Omega_kn} \]

where \( \Omega_k = k(2\pi/P) \), \( P \) is some integer (preferably a power of 2) such that \( P \geq L \), where \([0,L-1]\) is the time interval outside of which \( x[n] \) is zero, and \( k \) ranges from \(-P/2\) to \((P/2)-1\) (for even \( P \)).
Where do the $\Omega_k$ live?
e.g., for $P=6$ (even)

$\Omega_{-3}$ $\Omega_{-2}$ $\Omega_{-1}$ $\Omega_0$ $\Omega_1=\frac{2\pi}{P}$ $\Omega_2$ $\Omega_3=3\Omega_1$

$-\pi$ 0 $\pi$

$\exp(j\Omega_2)$ $\exp(j\Omega_1)$

$\exp(j\Omega_3) = \exp(j\Omega_{-3})$

1 $\exp(j\Omega_0)$

$\exp(j\Omega_{-2})$ $\exp(j\Omega_{-1})$
Fast Fourier Transform (FFT) to compute samples of the DTFT for signals of finite duration

\[ X(\Omega_k) = \sum_{m=0}^{P-1} x[m]e^{-j\Omega_km}, \quad x[n] = \frac{1}{P} \sum_{k=-P/2}^{(P/2)-1} X(\Omega_k)e^{j\Omega_kn} \]

where \( \Omega_k = k(2\pi/P) \), \( P \) is some integer (preferably a power of 2) such that \( P \geq L \), where \([0,L-1]\) is the time interval outside of which \( x[n] \) is zero, and \( k \) ranges from \(-P/2\) to \((P/2)-1\) (for even \( P \)).

Computing these sums directly involves \( O(P^2) \) operations – when \( P \) gets large, the computations get very slow…..

Happily, in 1965 Cooley and Tukey published a fast (divide-and-conquer) method for computing the Fourier transform (aka FFT, IFFT), rediscovering a technique known to Gauss. This method takes \( O(P \log P) \) operations.

\[ P = 1024, \quad P^2 = 1,048,576, \quad P \log P \approx 10,240 \]
Relation to DT Fourier Series

\[ x[n] = \frac{1}{P} \sum_{k=-P/2}^{(P/2)-1} X(\Omega_k) e^{j\Omega_k n} \]

- Evaluating this eqn. for \( n \) in \([0,P-1]\) recovers the original \( x[n] \) in this interval.

- Evaluating it for \( n \) outside this interval results in periodic replication of the values in \([0,P-1]\), producing a periodic signal \( x[n] \).

- So this eqn. is also called a DT Fourier Series (DTFS) for the periodic signal \( x[n] \). Notation: \( A_k = X(\Omega_k)/P \), Fourier coefficient. Use this relationship to make sense of Chapter 14 notes!
Spectral Content of Various Sounds

Speech spectrogram

http://en.wikipedia.org/wiki/Spectrogram

**Prof. Victor Zue was the first person to be able to read these visually!**

Dolphin sounds
Instantaneous Heart Rate

Instantaneous HR Signal

![Graph showing instantaneous heart rate over time](image)

- x(t) (beats/min)
- Time (secs)
Heart-Rate Power Spectral Density
$|X(\Omega)|^2$ averaged over many time-windows

- Breathing frequency: 0.18Hz
Relating Output Spectral Content to Input Spectral Content for an LTI System

\[ x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega)e^{j\Omega n} d\Omega \]

\[ y[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(\Omega)X(\Omega)e^{j\Omega n} d\Omega \]

\[ Y(\Omega) = H(\Omega)X(\Omega) \]

Compare with \( y[n]=(h*x)[n] \)

Again, convolution in time has mapped to multiplication in frequency.
Input/Output Behavior of LTI System in Frequency Domain

\[ x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega)e^{j\Omega n} d\Omega \]

\[ y[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(\Omega)X(\Omega)e^{j\Omega n} d\Omega \]

\[ Y(\Omega) = H(\Omega)X(\Omega) \]

Spectral content of output

Spectral content of input

Frequency response of system
Magnitude and Angle

\[ Y(\Omega) = H(\Omega)X(\Omega) \]

\[ |Y(\Omega)| = |H(\Omega)| \cdot |X(\Omega)| \]

and

\[ < Y(\Omega) = < H(\Omega) + < X(\Omega) \]
To Summarize:

1. A huge class of DT and CT signals can be written using Fourier transforms as a \textit{weighted linear combination of sinusoids} or (equivalently, but more compactly) \textit{complex exponentials}. The combinations can be \textit{discrete} $\Sigma$ or \textit{continuous} $\int$ (or both).

2. LTI systems act very simply on weighted linear combinations of sinusoids: \textit{superposition} of responses to each sinusoid, with the \textit{frequency response} determining the frequency-dependent scaling of magnitude, shifting in phase.