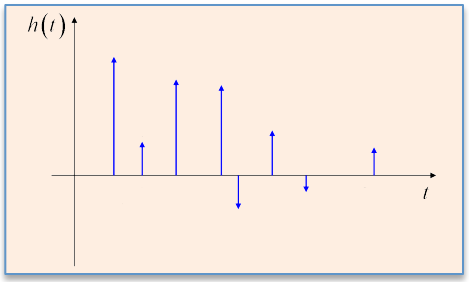
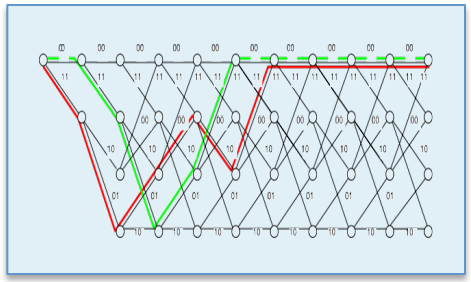
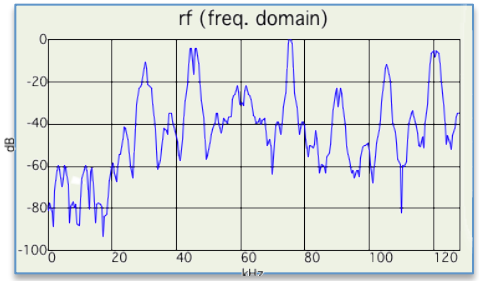
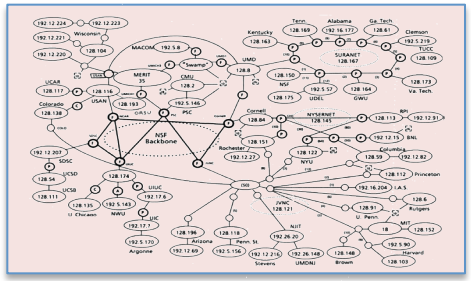








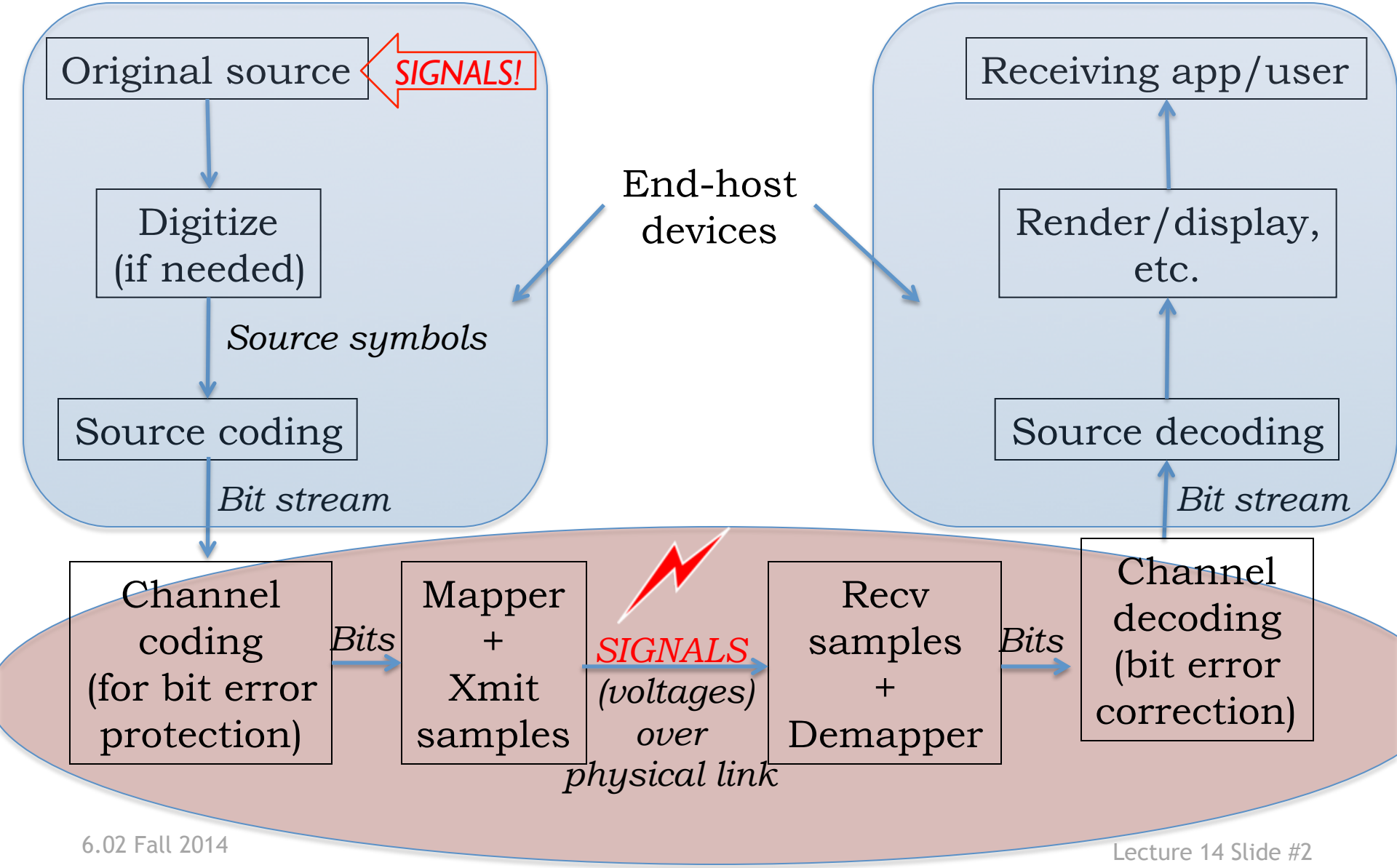
INTRODUCTION TO EECS II

DIGITAL COMMUNICATION SYSTEMS

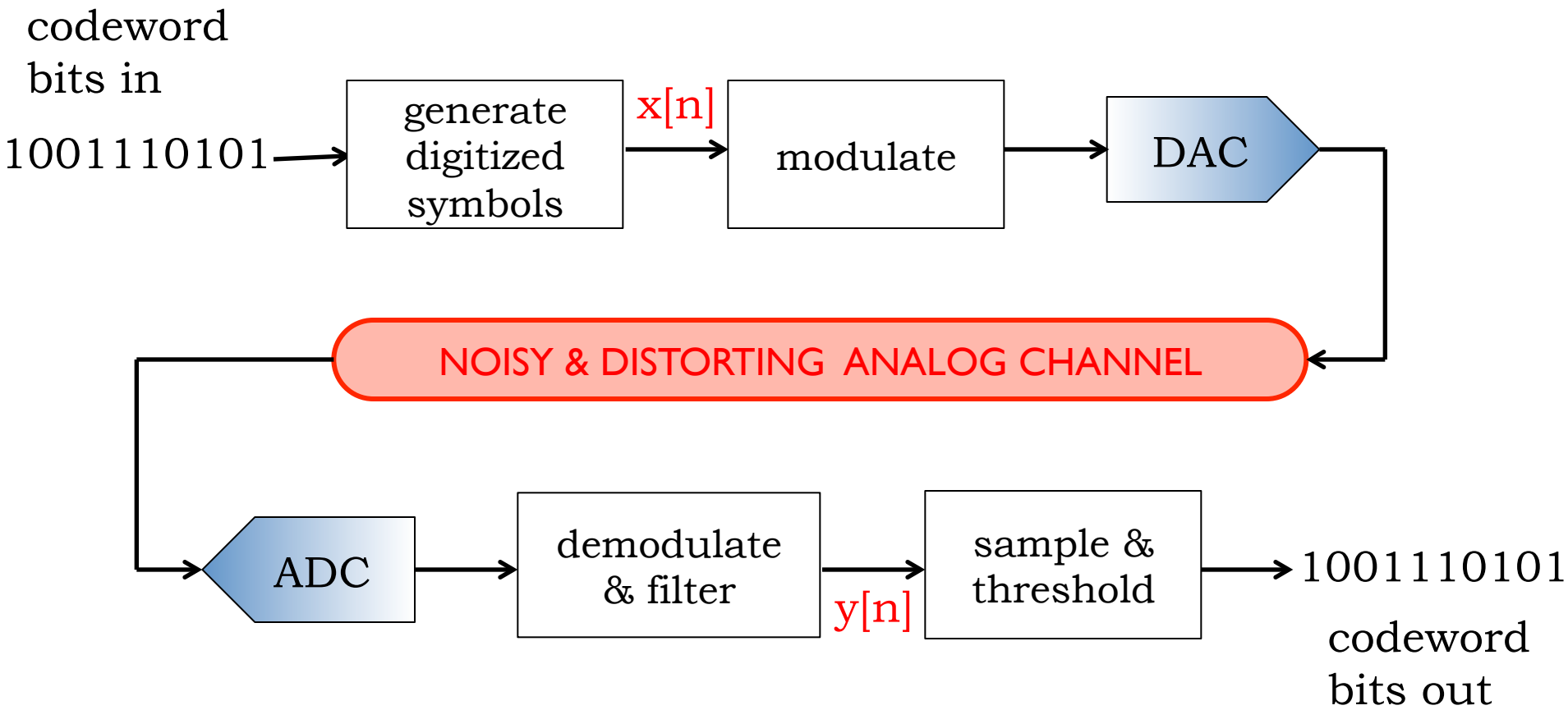
6.02 Fall 2014
Lecture #14

- Filtering
- Spectral content of signals via the DTFT

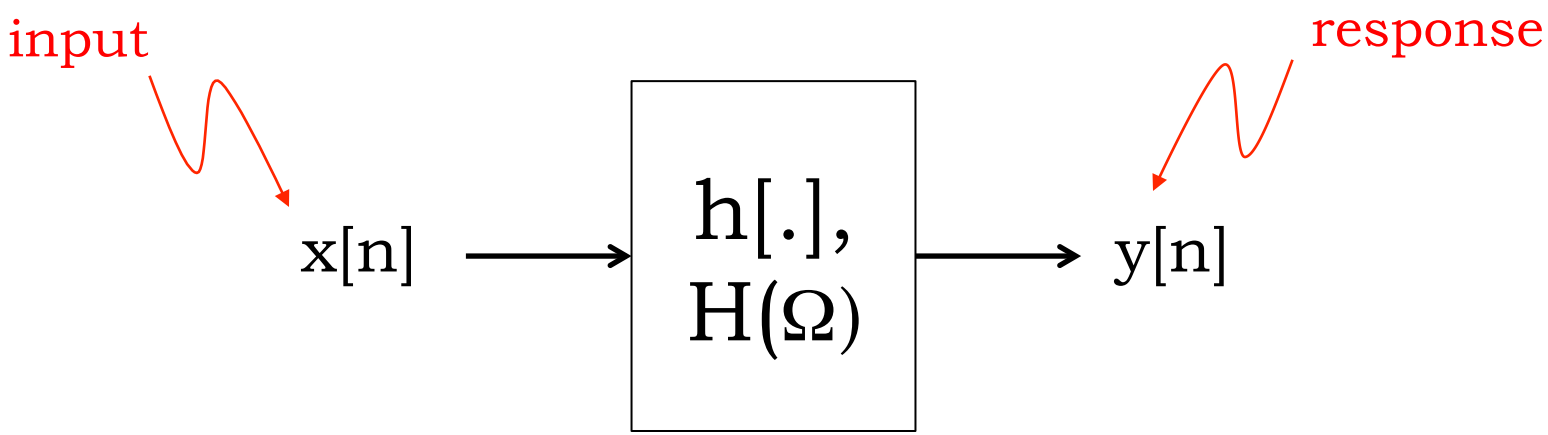
Single Link Communication Model



Modeling The Baseband Channel



The LTI Baseband** Channel Model



Frequency response H(Ω) describes the response to sinusoids (or equivalently, to complex exponentials, i.e., "cisoids")

Similar math for LTI filtering of y[n] for downstream processing

---

**From before the modulator till after the demodulator & filter

A Deeper Reason for Interest in Sinusoidal Inputs

- General inputs x[.] can be written as linear combinations of sinusoids (or complex exponentials)

- Each input sinusoidal component is mapped via the frequency response $H(\Omega)$ to its corresponding sinusoidal output component

- Superposition of these output components yields the general response y[.]

We'll see this in today's lecture.

Determining h[n] from H(Ω)

$$H(\Omega) = \sum_m h[m] e^{-j\Omega m}$$

Multiply both sides by $e^{j\Omega n}$ and integrate over a (contiguous) 2π interval. Only one term survives!

$$\int_{<2\pi>} H(\Omega) e^{j\Omega n} d\Omega = \int_{<2\pi>} \sum_m h[m] e^{-j\Omega(m-n)} d\Omega$$

$$= 2\pi \cdot h[n]$$

$$\Rightarrow \quad h[n] = \frac{1}{2\pi} \int_{<2\pi>} H(\Omega) e^{j\Omega n} d\Omega$$

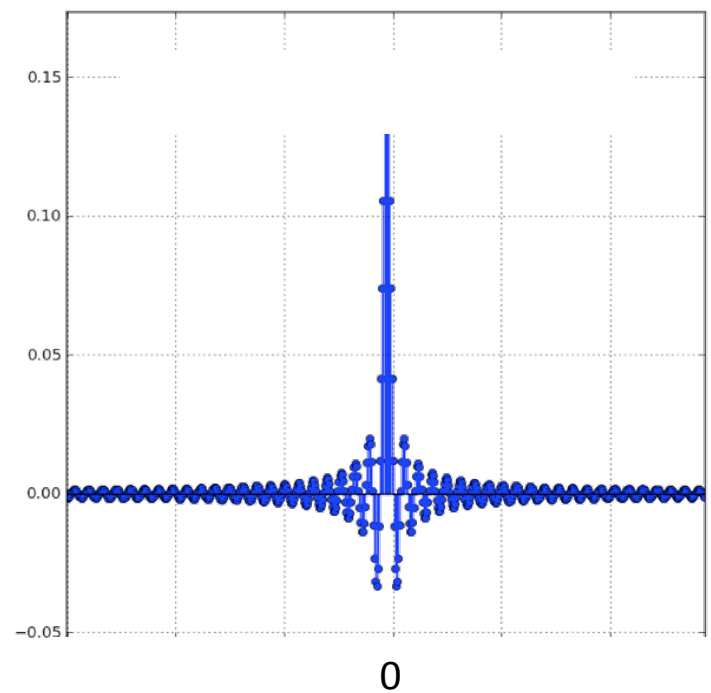
Design ideal lowpass filter with cutoff frequency Ω_C and $H(\Omega)=1$ in passband

$$h[n] = \frac{1}{2\pi} \int_{<2\pi>} H(\Omega) e^{j\Omega n} d\Omega$$

$$= \frac{1}{2\pi} \int_{-\Omega_C}^{\Omega_C} 1 \cdot e^{j\Omega n} d\Omega$$

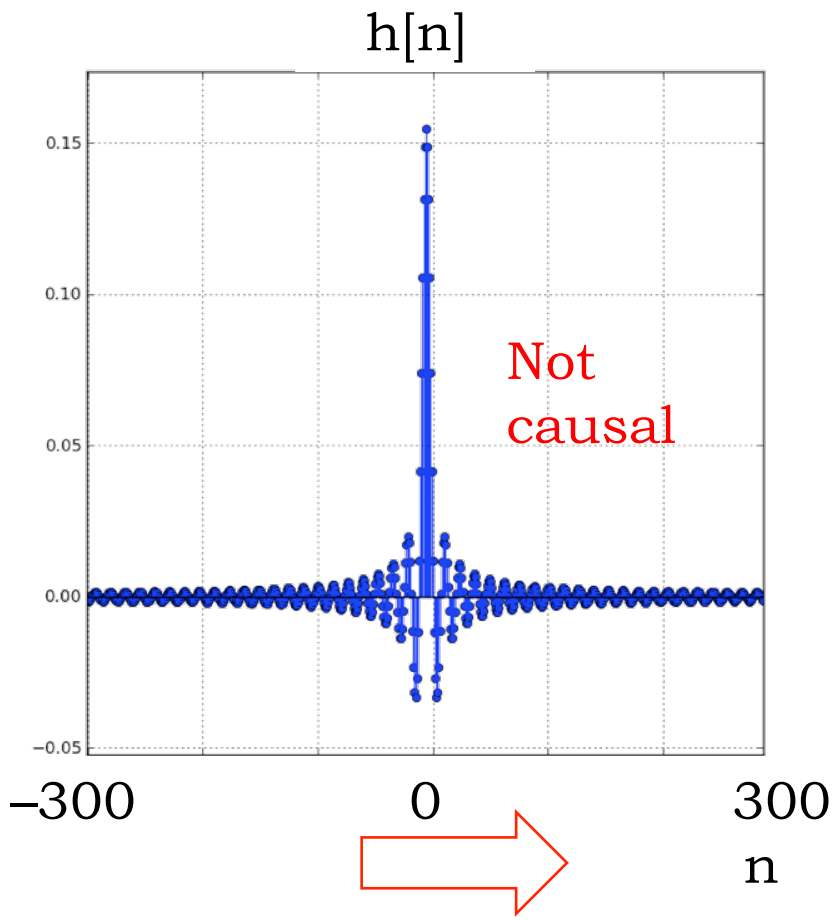
$$= \frac{\sin(\Omega_C n)}{\pi n} \quad , \quad n \neq 0$$

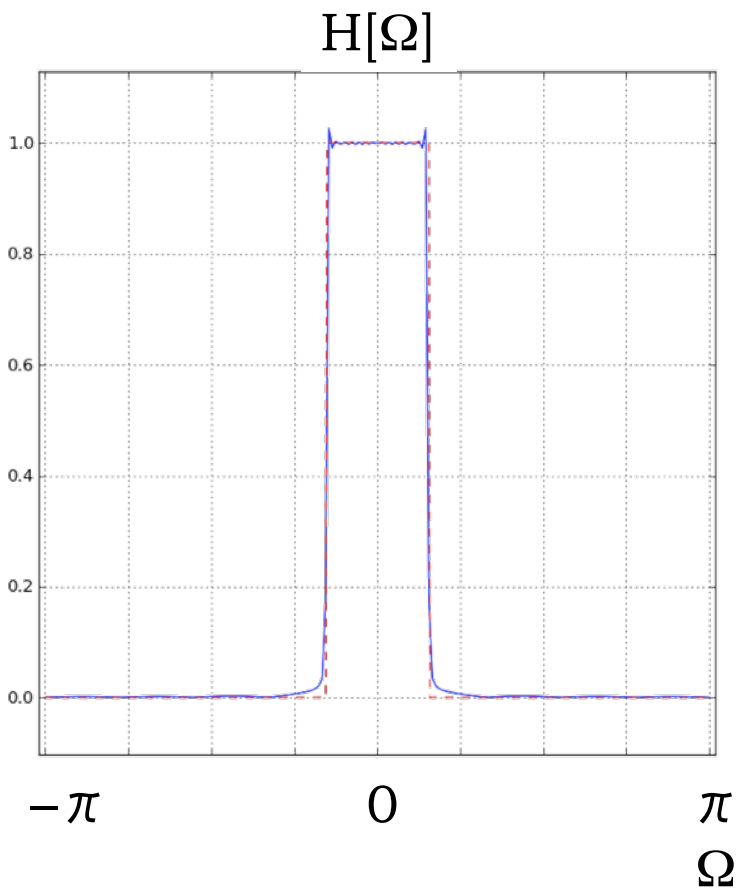
$$= \Omega_C / \pi \quad , \quad n = 0$$



DT "sinc" function
(extends to ±∞ in time, falls off only as 1/n))

Approximating an ideal lowpass filter

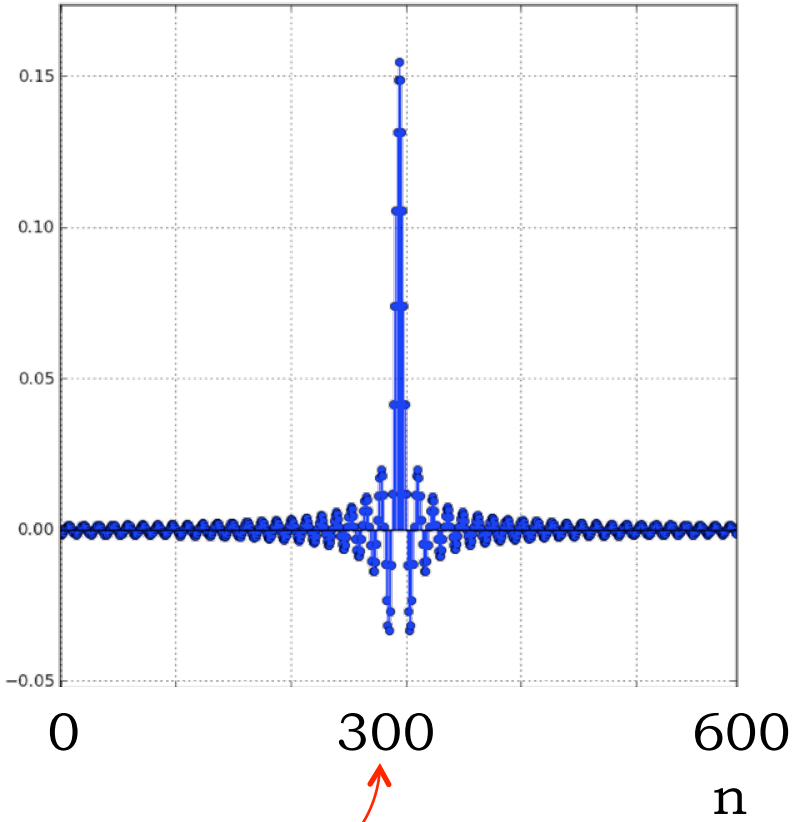




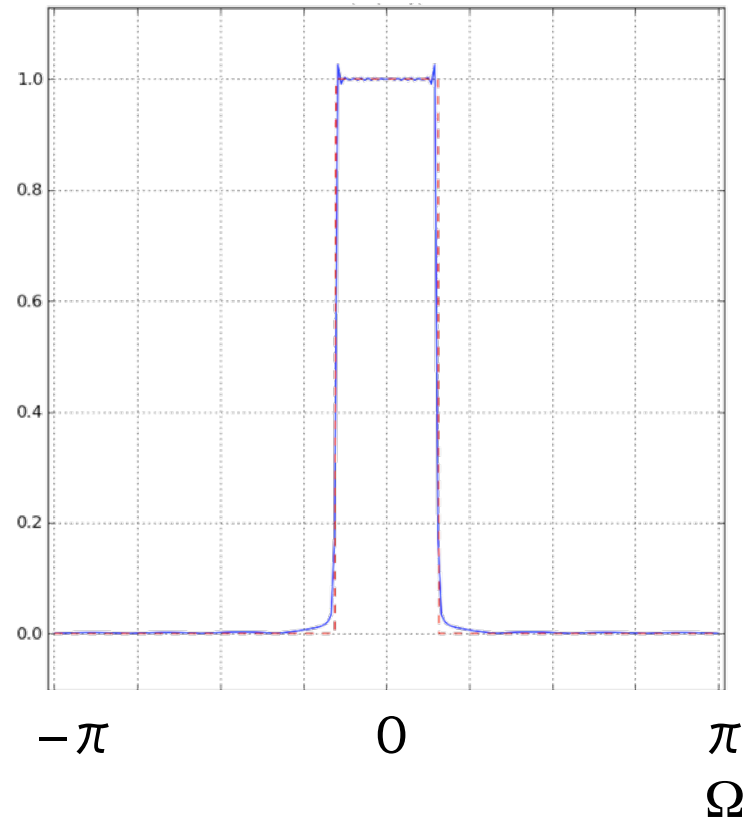
Idea: shift h[n] right to get causal LTI system.
Will the result still be a lowpass filter?

Causal approximation to ideal lowpass filter

$h_C[n] = h[n-300]$

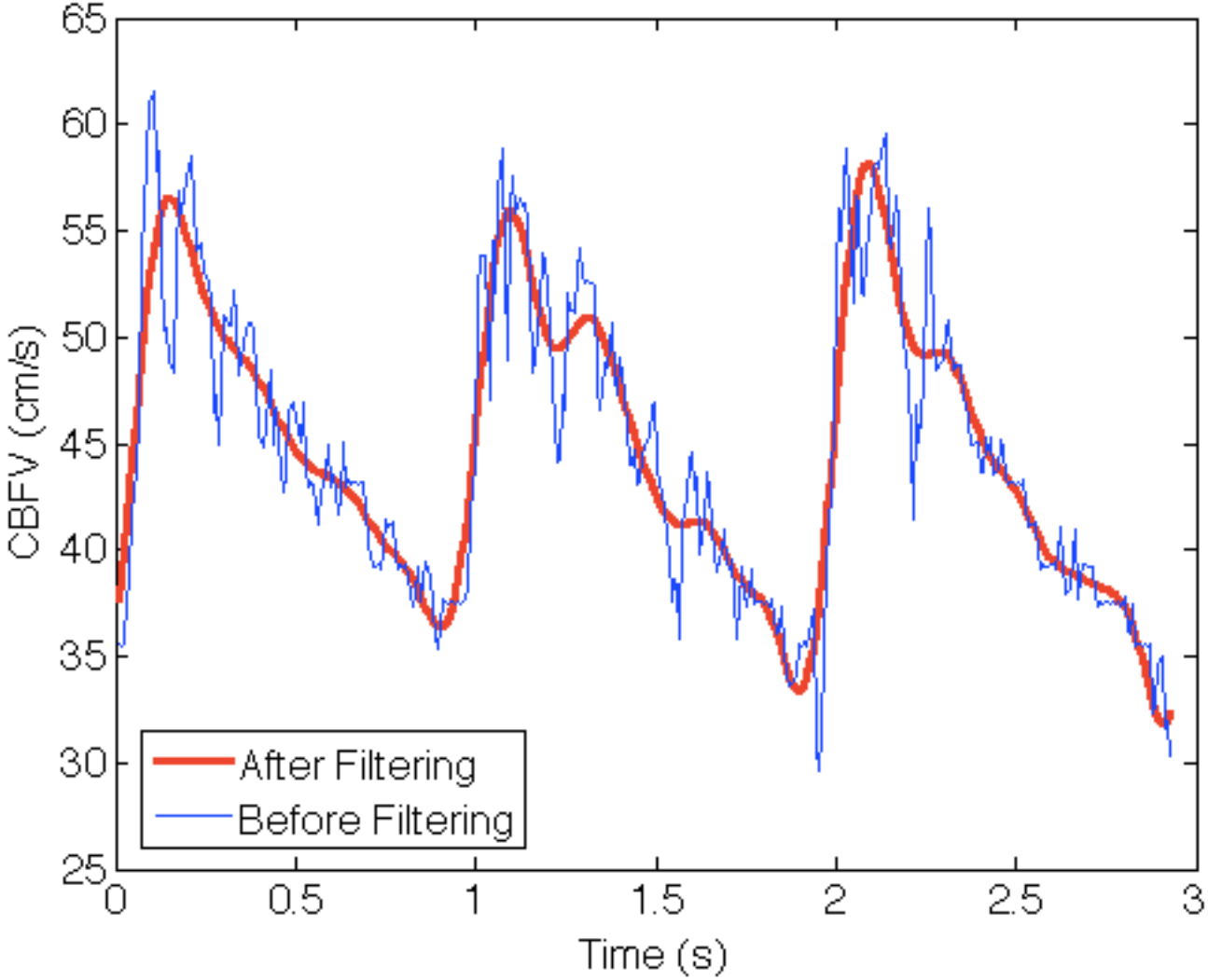


$|H_C[\Omega]|$



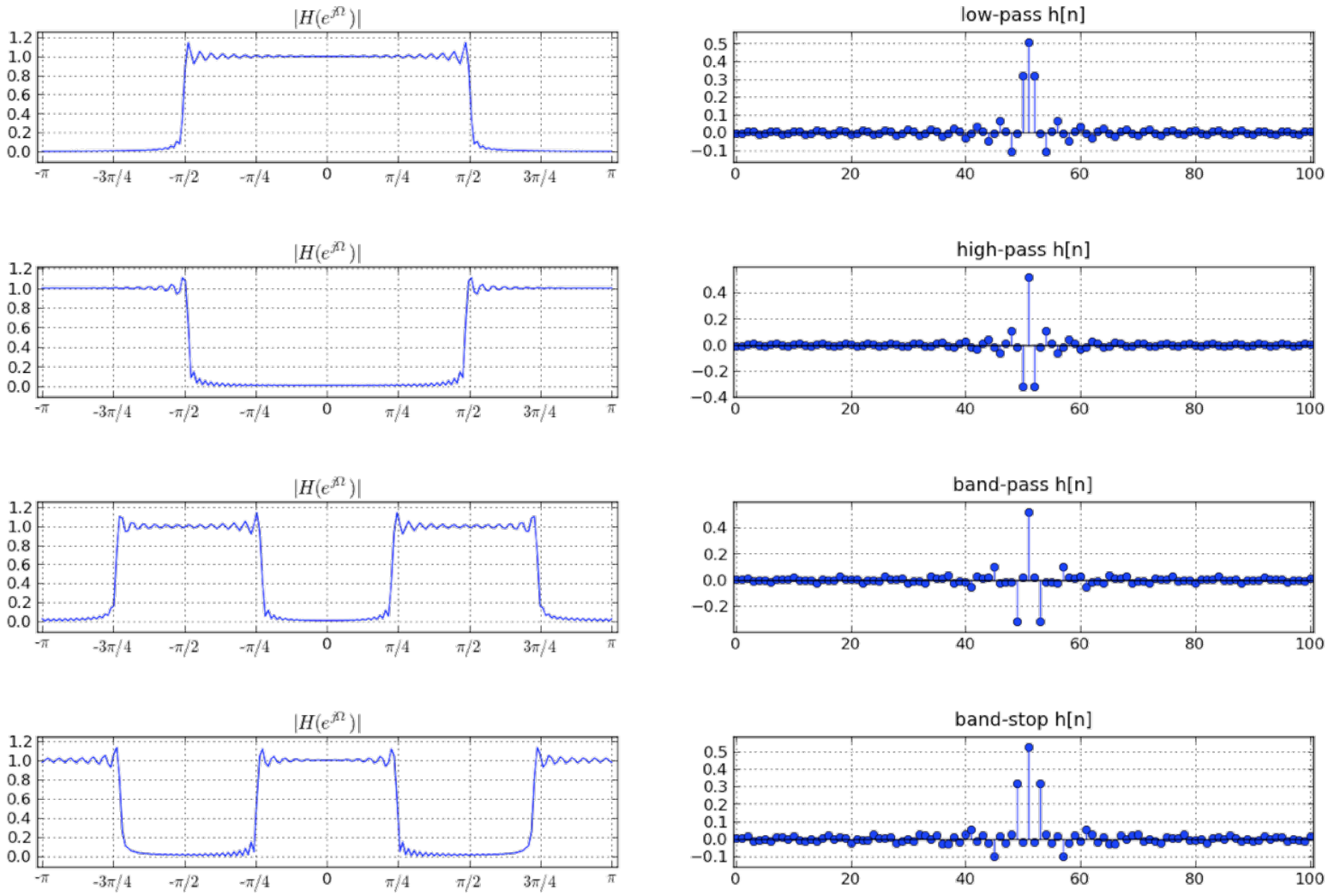
Determine $<H_C(\Omega)$

Filtering



Lowpass filtering (10 Hz cutoff) of blood flow velocity in middle cerebral artery, measured using transcranial Doppler ultrasound

Useful Filters



DT Fourier Transform (DTFT) for Spectral Representation of General x[n]

If we can write

$$h[n] = \frac{1}{2\pi} \int_{<2\pi>} H(\Omega) e^{j\Omega n} d\Omega \qquad \text{where} \qquad H(\Omega) = \sum_m h[n] e^{-j\Omega m}$$

Any contiguous interval of length 2π

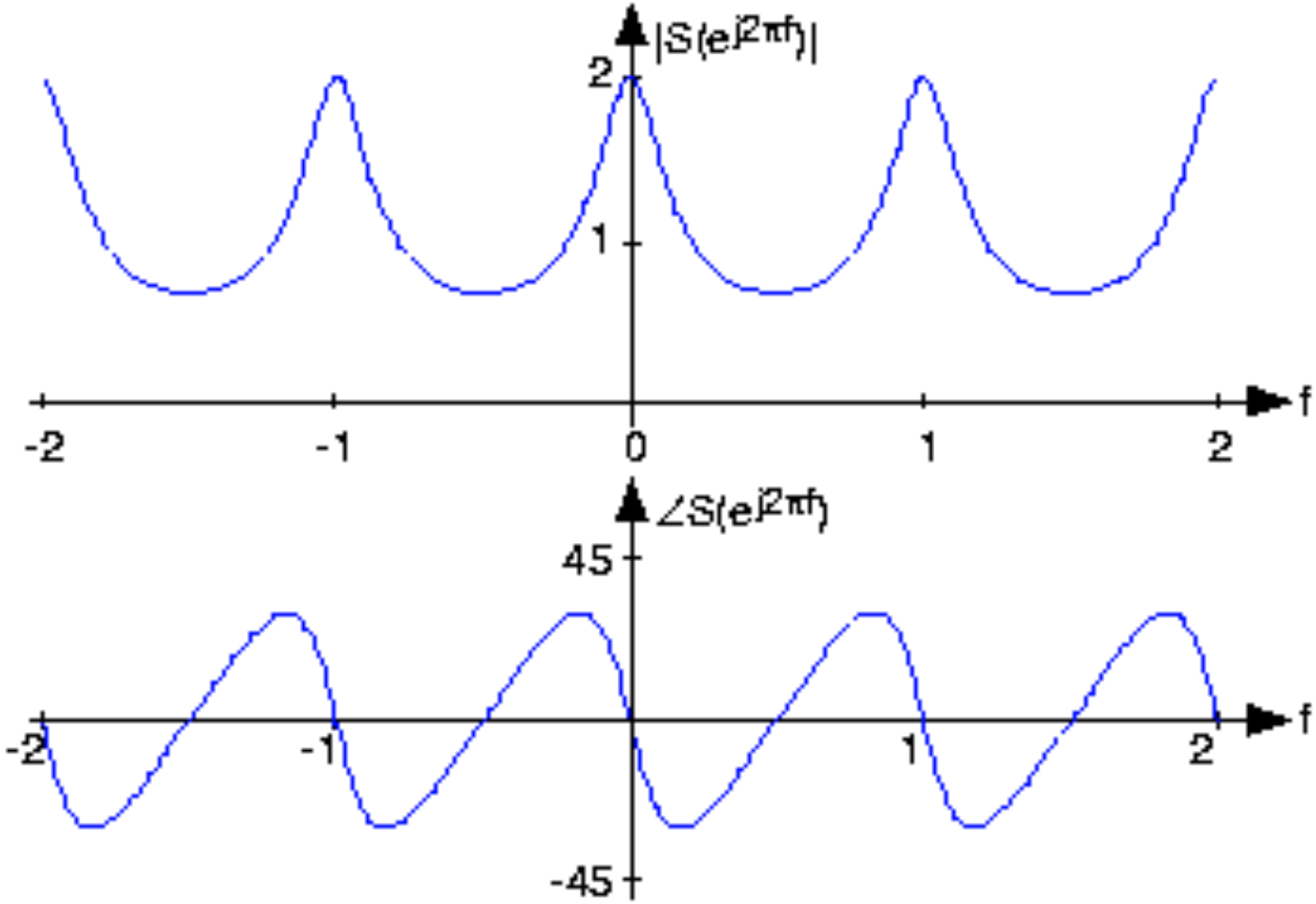
then we can write

$$x[n] = \frac{1}{2\pi} \int_{<2\pi>} X(\Omega) e^{j\Omega n} d\Omega \qquad \text{where} \qquad X(\Omega) = \sum_m x[n] e^{-j\Omega m}$$

This Fourier representation expresses x[n] as a weighted combination of $e^{j\Omega n}$ for all Ω in $[-\pi, \pi]$.

$X(\Omega_o)d\Omega$ is the **spectral content** of x[n] in the frequency interval $[\Omega_o, \Omega_o + d\Omega]$

The spectrum of the exponential signal $(0.5)^n u[n]$ is shown over the frequency range $\Omega = 2\pi f$ in $[-4\pi, 4\pi]$, The angle has units of degrees.



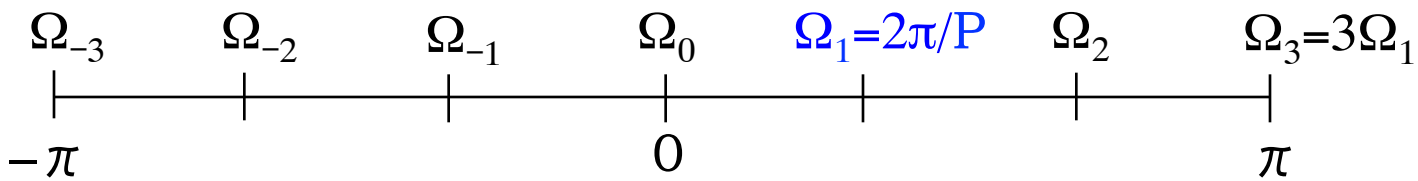
http://cnx.org/content/m0524/latest/

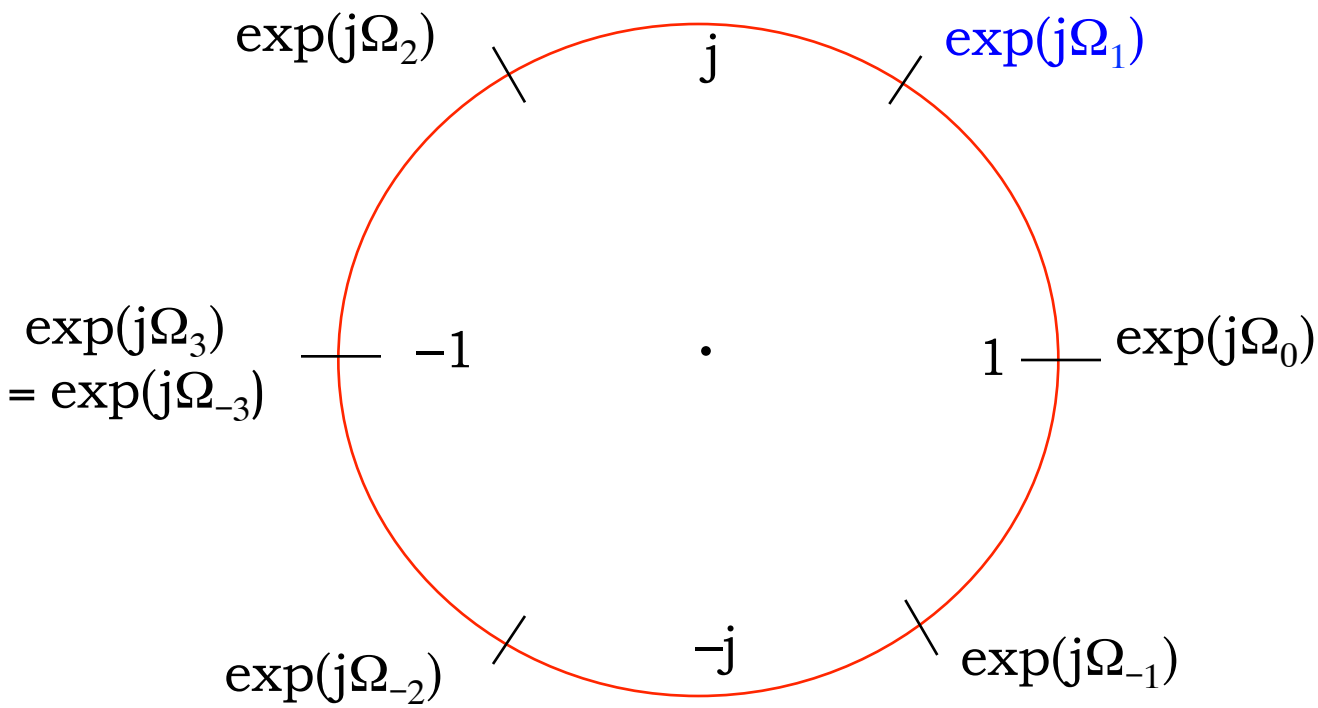
Fast Fourier Transform (FFT) to compute samples of the DTFT for signals of finite duration

$$X(\Omega_k) = \sum_{m=0}^{P-1} x[m] e^{-j\Omega_k m}, \qquad x[n] = \frac{1}{P} \sum_{k=-P/2}^{(P/2)-1} X(\Omega_k) e^{j\Omega_k n}$$

where $\Omega_k = k(2\pi/P)$, P is some integer (preferably a power of 2) such that $P \geq L$, where [0,L-1] is the time interval outside of which x[n] is zero, and k ranges from –P/2 to (P/2)–1 (for even P).

Where do the Ω_k live? e.g., for P=6 (even)





Fast Fourier Transform (FFT) to compute samples of the DTFT for signals of finite duration

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where $\Omega_k = k(2\pi/P)$, P is some integer (preferably a power of 2) such that $P \geq L$, where [0,L-1] is the time interval outside of which x[n] is zero, and k ranges from –P/2 to (P/2)–1 (for even P).

Computing these sums directly involves $O(P^2)$ operations – when P gets large, the computations get very s l o w....

Happily, in 1965 Cooley and Tukey published a fast (divide-and- conquer) method for computing the Fourier transform (aka **FFT**, IFFT), rediscovering a technique known to Gauss. This method takes O(P log P) operations.

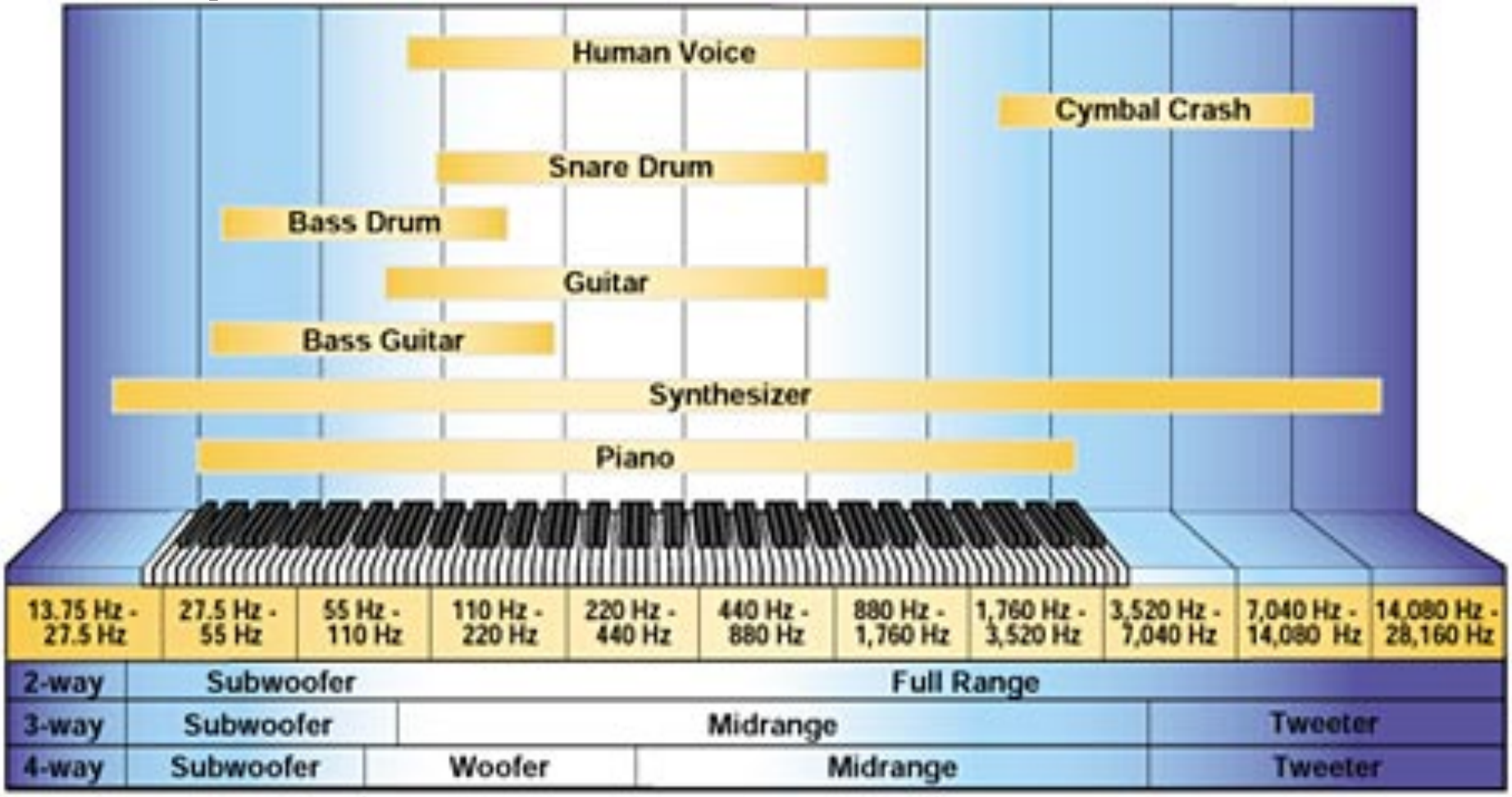
$P = 1024$, $P^2 = 1{,}048{,}576$, $P \log P \approx 10{,}240$

Relation to DT Fourier Series

$$x[n] = \frac{1}{P}\sum_{k=-P/2}^{(P/2)-1} X(\Omega_k)e^{j\Omega_k n}$$

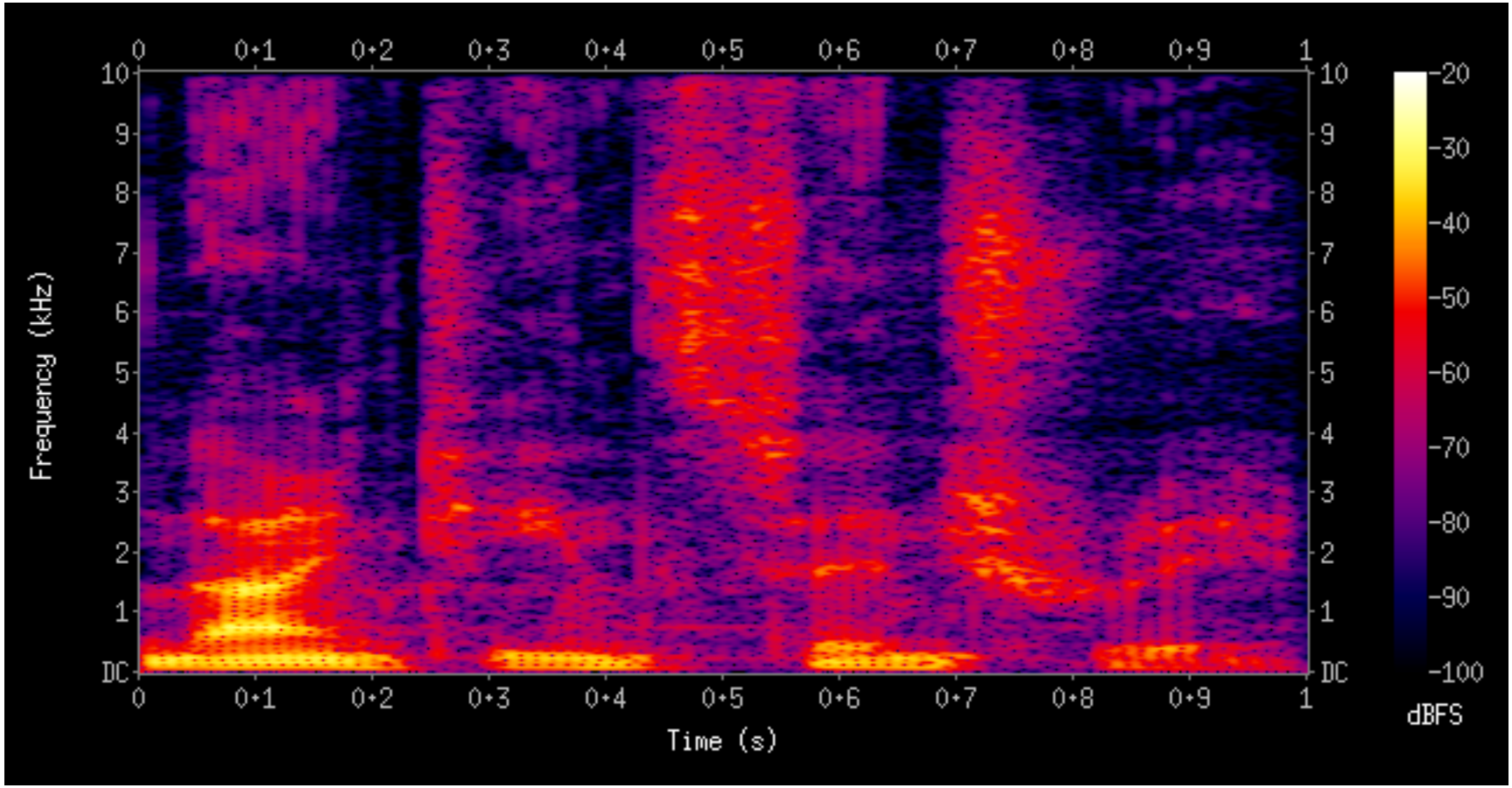
- Evaluating this eqn. for n in [0,P-1] recovers the original x[n] in this interval
- Evaluating it for n outside this interval results in periodic replication of the values in [0,P-1], producing a periodic signal $\boldsymbol{x}$[n]
- So this eqn. is also called a DT Fourier Series (DTFS) for the periodic signal $\boldsymbol{x}$[n]. Notation: $A_k = X(\Omega_k)/P$, Fourier c*oefficient*.
  Use this relationship to make sense of Chapter 14 notes!

Spectral Content of Various Sounds



http://forum.blu-ray.com/showthread.php?t=150915

Speech spectrogram

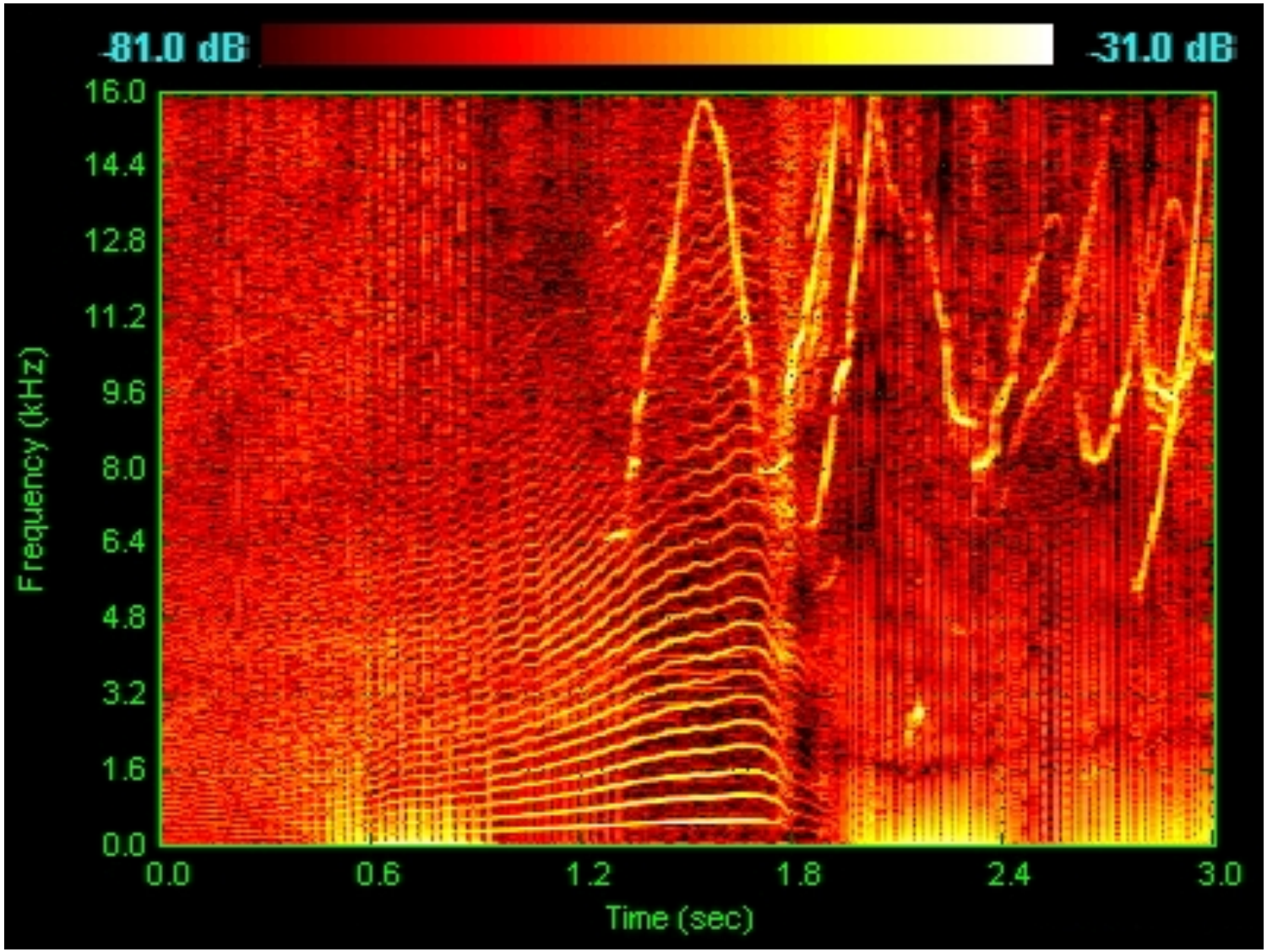


http://en.wikipedia.org/wiki/Spectrogram

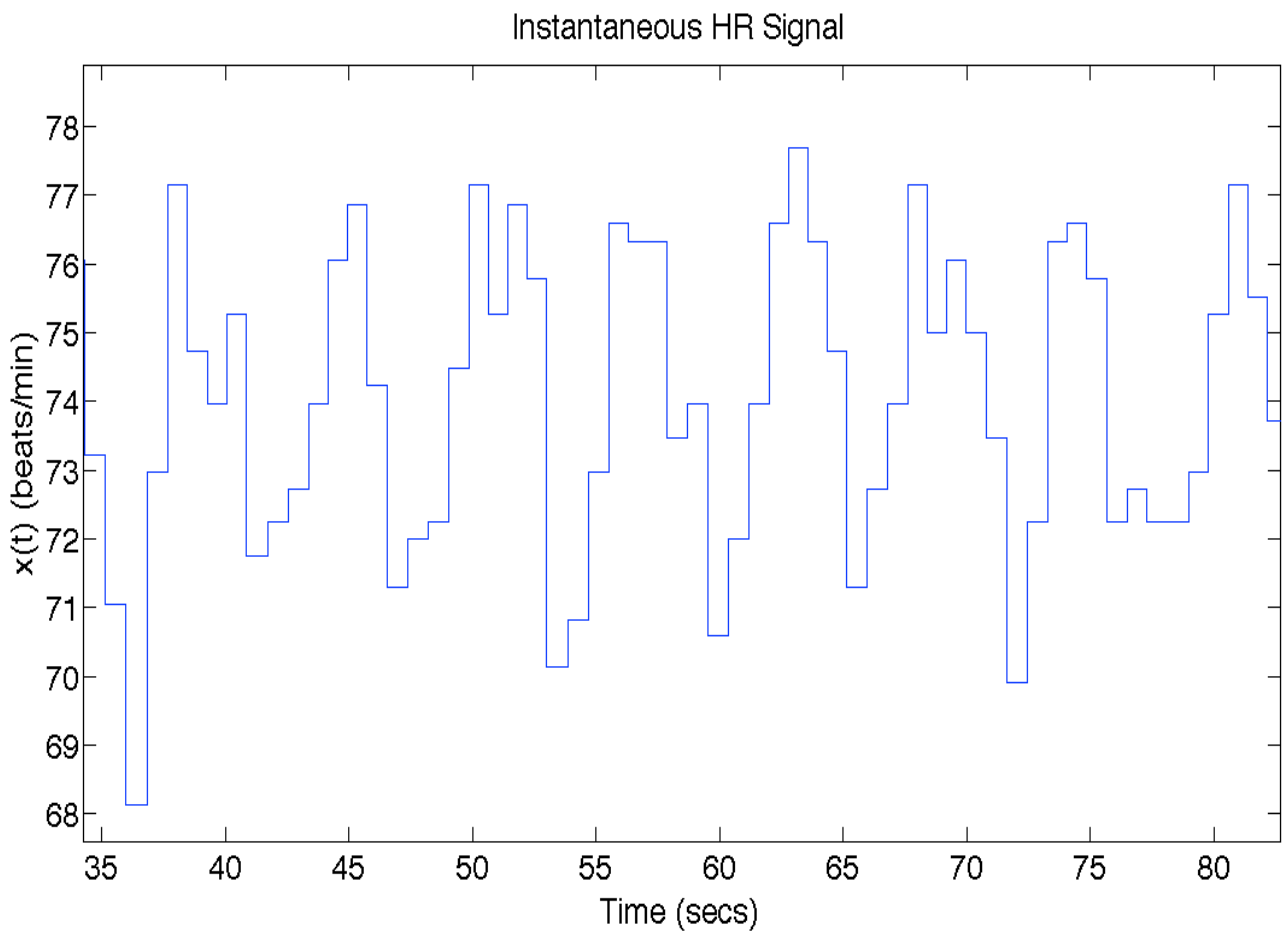
**Prof. Victor Zue was the first person to be able to read these visually!

http://www.okawa-foundation.or.jp/en/activities/prize/data/2012_evi.pdf

Dolphin sounds

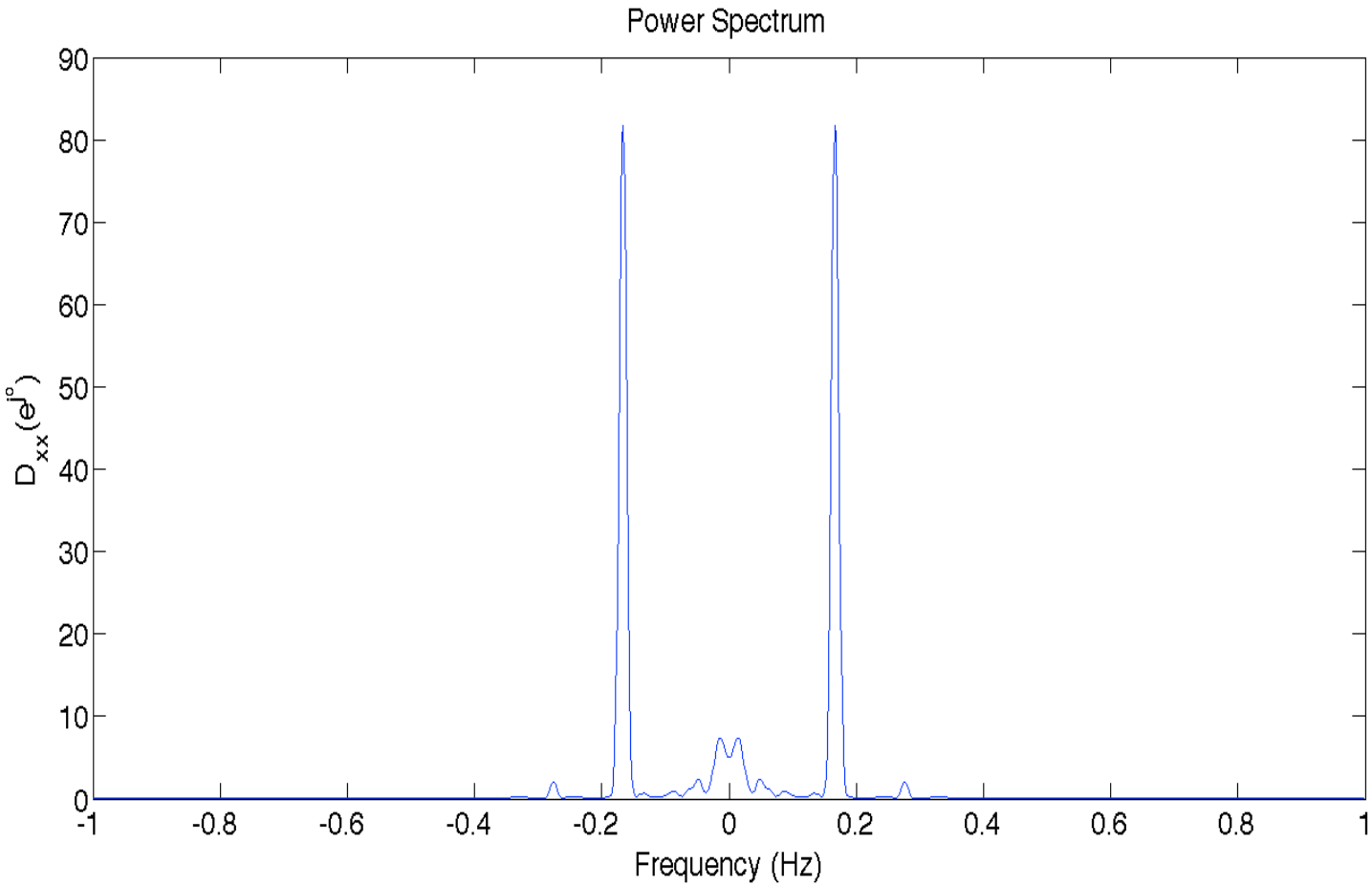


Instantaneous Heart Rate



Heart-Rate Power Spectral Density $|X(\Omega)|^2$ averaged over many time-windows

- Breathing frequency: 0.18Hz



Relating Output Spectral Content to Input Spectral Content for an LTI System

$$x[n] = \frac{1}{2\pi} \int_{<2\pi>} X(\Omega) e^{j\Omega n} d\Omega$$

$$\rightarrow \boxed{H(\Omega)} \rightarrow$$

$$y[n] = \frac{1}{2\pi} \int_{<2\pi>} H(\Omega) X(\Omega) e^{j\Omega n} d\Omega$$

$$y[n] = \frac{1}{2\pi} \int_{<2\pi>} Y(\Omega) e^{j\Omega n} d\Omega$$

$$Y(\Omega) = H(\Omega) X(\Omega)$$

Compare with y[n]=(h*x)[n]

Again, convolution in time
has mapped to
multiplication in frequency

Input/Output Behavior of LTI System in Frequency Domain

$$x[n] = \frac{1}{2\pi} \int_{<2\pi>} X(\Omega) e^{j\Omega n} d\Omega$$

$$\rightarrow \boxed{H(\Omega)} \rightarrow$$

$$y[n] = \frac{1}{2\pi} \int_{<2\pi>} H(\Omega) X(\Omega) e^{j\Omega n} d\Omega$$

$$y[n] = \frac{1}{2\pi} \int_{<2\pi>} Y(\Omega) e^{j\Omega n} d\Omega$$

$$Y(\Omega) = H(\Omega) X(\Omega)$$

Spectral content of output

Frequency response of system

Spectral content of input

Magnitude and Angle

$$Y(\Omega) = H(\Omega)X(\Omega)$$

$$\Updownarrow$$

$$|Y(\Omega)| = |H(\Omega)| . |X(\Omega)|$$

and

$$< Y(\Omega) = < H(\Omega) + < X(\Omega)$$

To Summarize:

1. A huge class of DT and CT signals can be written ---
using **Fourier** transforms --- as a **weighted linear combination of sinusoids**
or (equivalently, but more compactly) complex exponentials.
The combinations can be discrete ∑ or continuous ∫ (or both).

2. **LTI** systems act very simply on weighted linear combinations of sinusoids:
superposition of responses to each sinusoid, with the
frequency response determining the frequency-dependent
scaling of magnitude, shifting in phase.