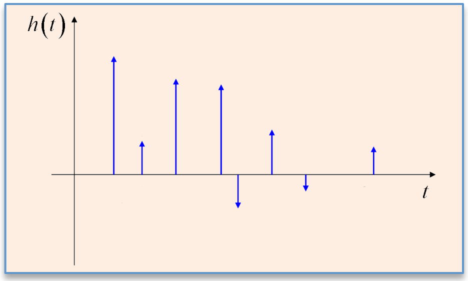
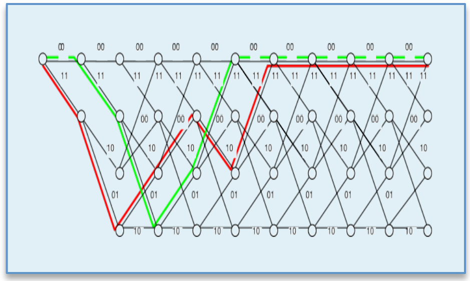
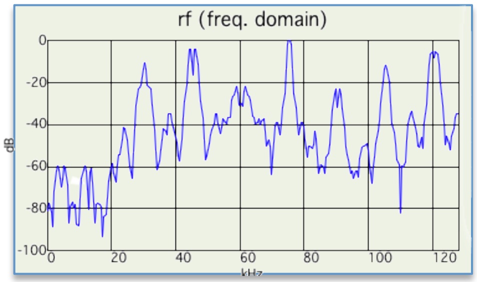
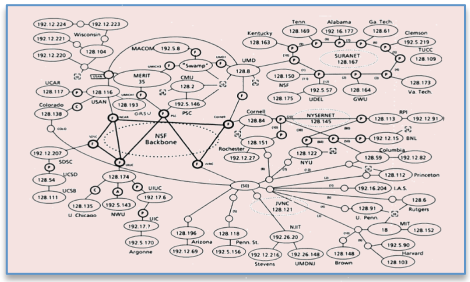








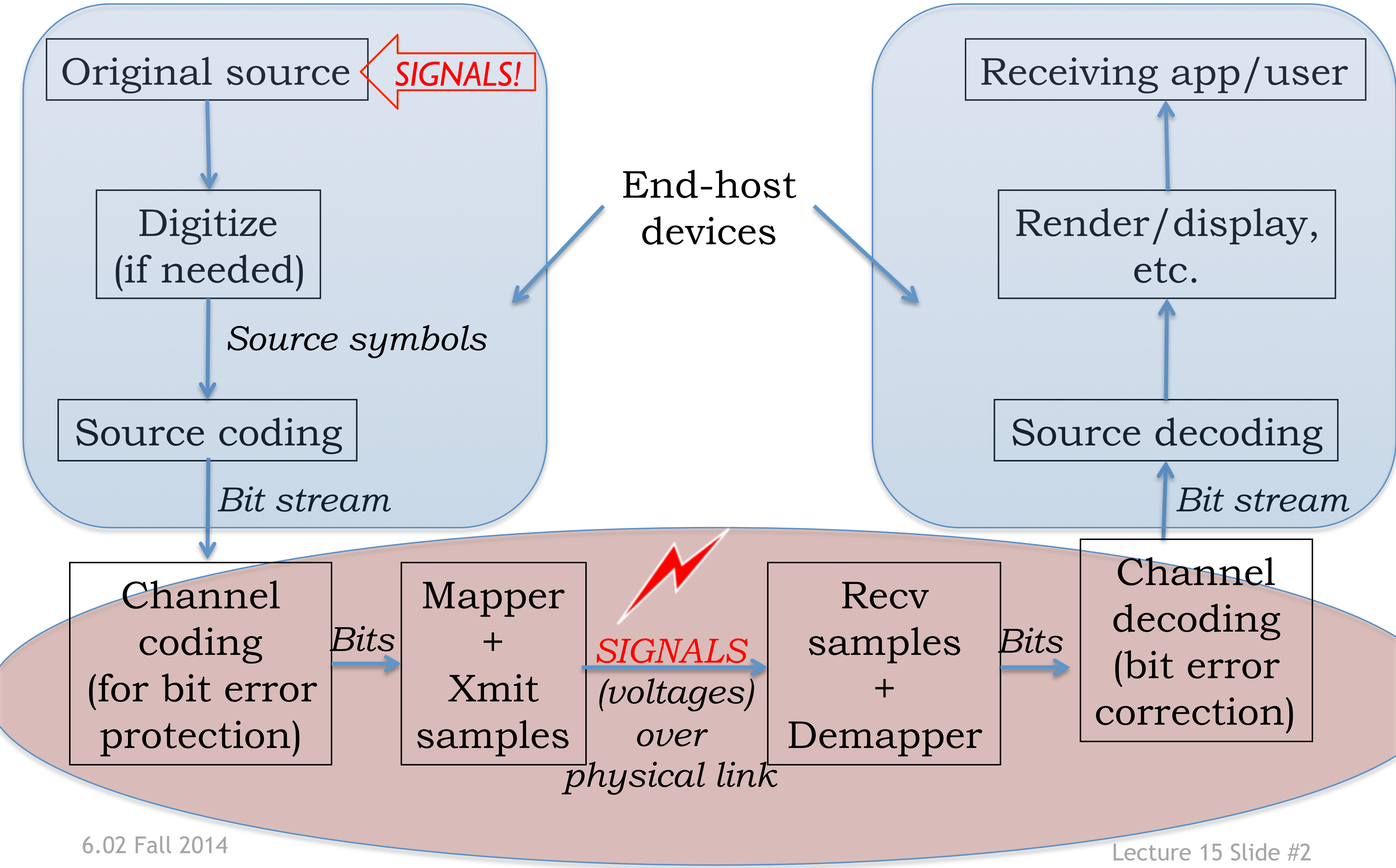
INTRODUCTION TO EECS II

DIGITAL COMMUNICATION SYSTEMS

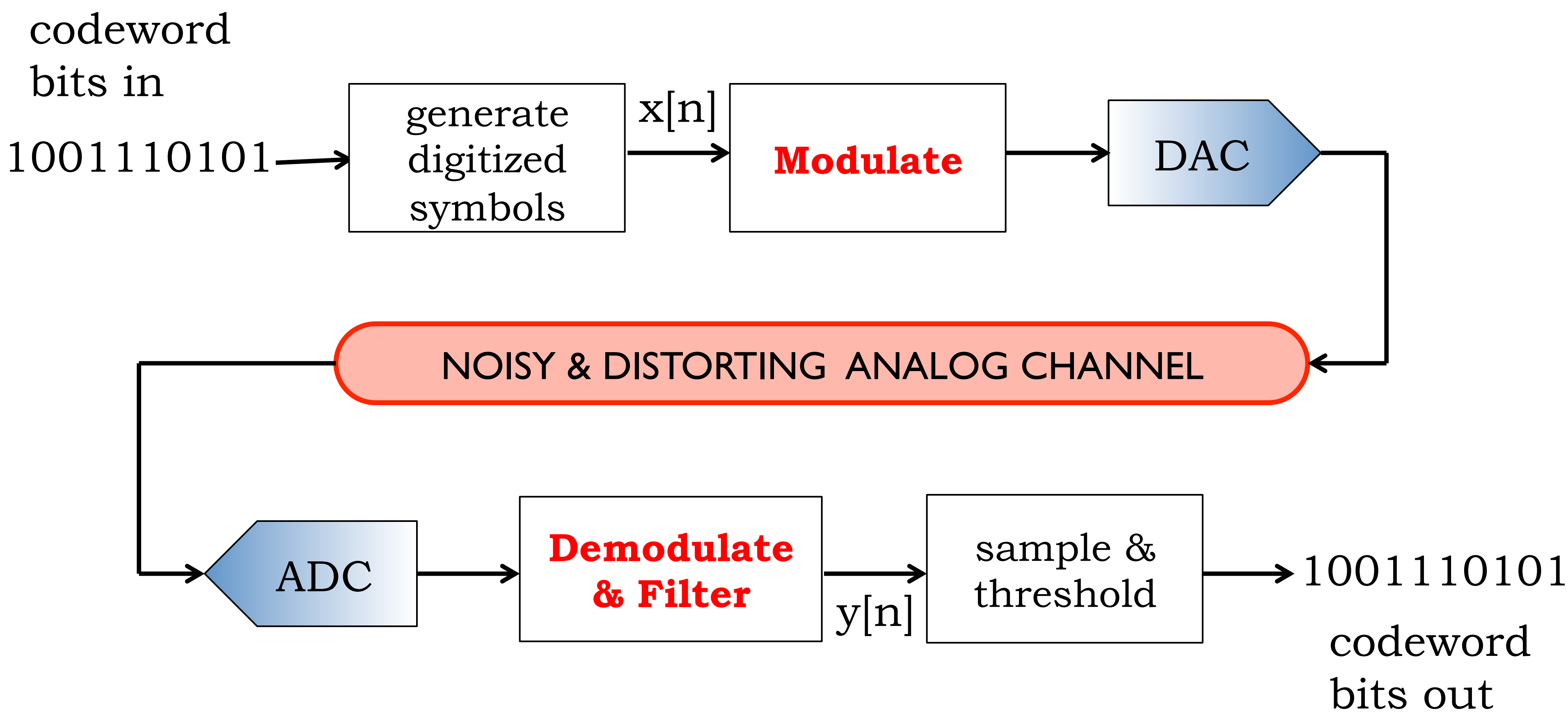
6.02 Fall 2014
Lecture #15

- Spectral Content via DTFT
- Modulation
- Intro to Demodulation

Single Link Communication Model



Modeling The Baseband Channel



DT Fourier Transform (DTFT) for Spectral Representation of General x[n]

$$x[n] = \frac{1}{2\pi} \int_{<2\pi>} X(\Omega) e^{j\Omega n} d\Omega$$

where

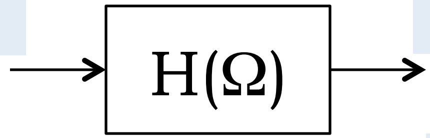
$$X(\Omega) = \sum_m x[n] e^{-j\Omega m}$$

This Fourier representation expresses x[n] as a weighted combination of $e^{j\Omega n}$ for **all** Ω in $[-\pi, \pi]$.

$X(\Omega_o)d\Omega$ is the **spectral content** of x[n] in the frequency interval $[\Omega_o, \Omega_o + d\Omega]$

Relating Output Spectral Content to Input Spectral Content for an LTI System

$$x[n] = \frac{1}{2\pi} \int_{<2\pi>} X(\Omega) e^{j\Omega n} d\Omega$$



$$y[n] = \frac{1}{2\pi} \int_{<2\pi>} H(\Omega) X(\Omega) e^{j\Omega n} d\Omega$$

$$y[n] = \frac{1}{2\pi} \int_{<2\pi>} Y(\Omega) e^{j\Omega n} d\Omega$$

$$Y(\Omega) = H(\Omega) X(\Omega)$$

Compare with y[n]=(h*x)[n]

Again, convolution in time has mapped to multiplication in frequency

Magnitude and Angle

$$Y(\Omega) = H(\Omega)X(\Omega)$$

$$\Updownarrow$$

$$|Y(\Omega)| = |H(\Omega)| . |X(\Omega)|$$

and

$$< Y(\Omega) = < H(\Omega) + < X(\Omega)$$

A Special Case: Sinusoidal Signals

We can handle sinusoids via the DTFT, but that requires working with Dirac impulses in frequency (because sinusoids are infinitely concentrated in frequency):

$$e^{j\Omega_o n} \leftrightarrow 2\pi\delta(\Omega - \Omega_o)$$

$$\cos(\Omega_o n) \leftrightarrow \pi[\delta(\Omega - \Omega_o) + \delta(\Omega + \Omega_o)]$$

$$\sin(\Omega_o n) \leftrightarrow j\pi[\delta(\Omega + \Omega_o) - \delta(\Omega - \Omega_o)]$$

So in 6.02 we instead deal with sinusoidal inputs using the following (and by now very familiar) fact:

$$A_o \cos(\Omega_o n + \theta_o) \rightarrow \mathbf{H}(\Omega) \rightarrow |H(\Omega_o)| A_o \cos\Big(\Omega_o n + \theta_o + \angle H(\Omega_o)\Big)$$

Fast Fourier Transform (FFT) to compute samples of the DTFT for signals of finite duration

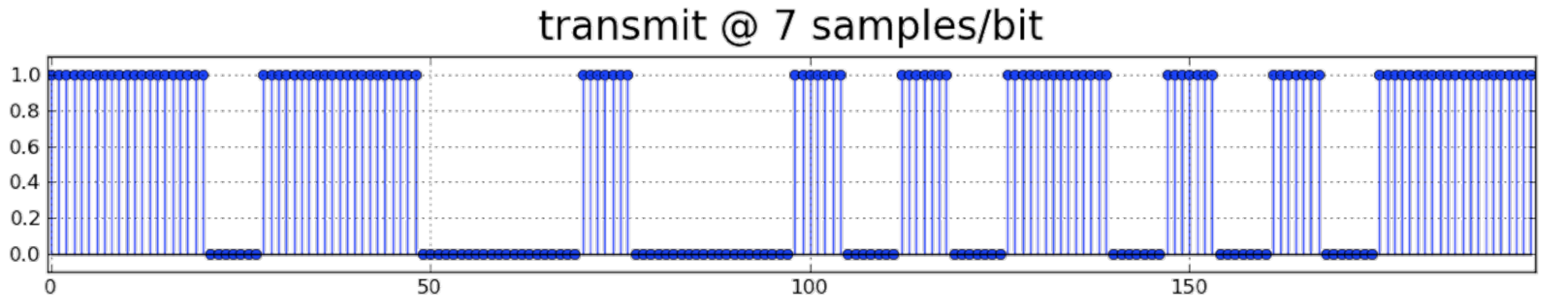
$$X(\Omega_k) = \sum_{m=0}^{P-1} x[m] e^{-j\Omega_k m}, \qquad x[n] = \frac{1}{P} \sum_{k=-P/2}^{(P/2)-1} X(\Omega_k) e^{j\Omega_k n}$$

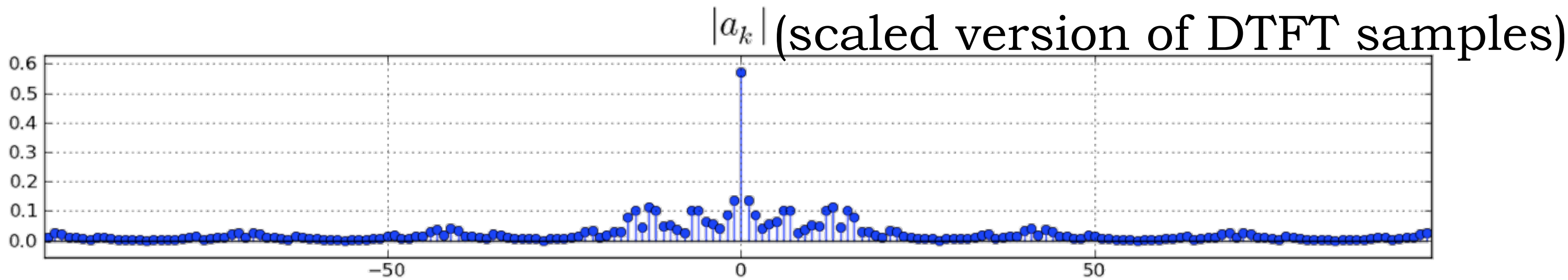
where $\Omega_k = k(2\pi/P)$, P is some integer (preferably a power of 2) such that P ≥ L, where [0,L-1] is the time interval outside of which x[n] is zero, and k ranges from –P/2 to (P/2)–1 (for even P).

Sometimes rewritten in terms of a_k or $A_k = X(\Omega_k)/P$, Fourier *coefficient.*

Use $X(\Omega_k) = P.A_k$ or $P.a_k$ to make sense of Chapter 14 notes!

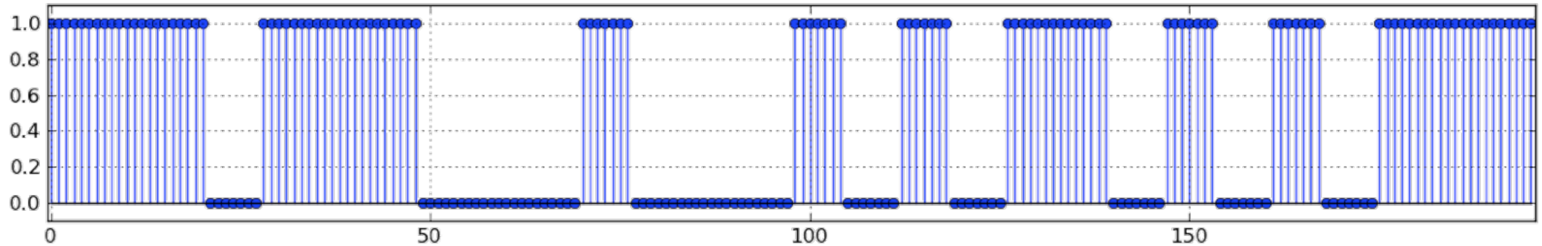
Spectrum of Digital Transmissions



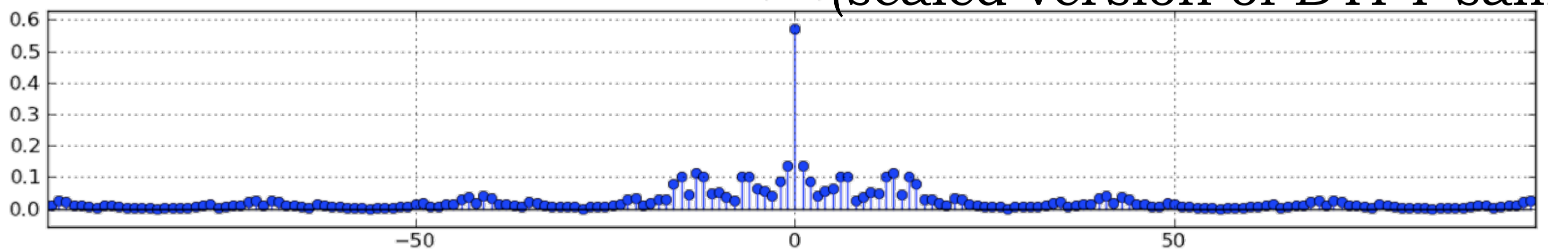


Spectrum of Digital Transmissions

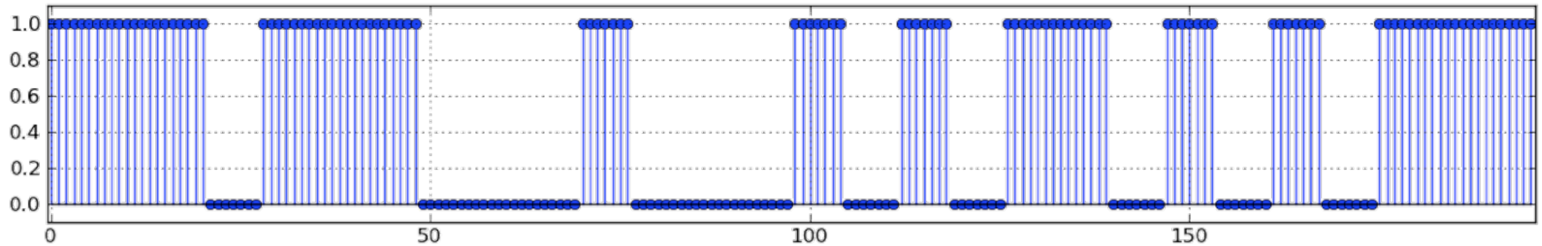
transmit @ 7 samples/bit



$|a_k|$ (scaled version of DTFT samples)



$x[n]$ synthesized from a_k



Observations on previous figure

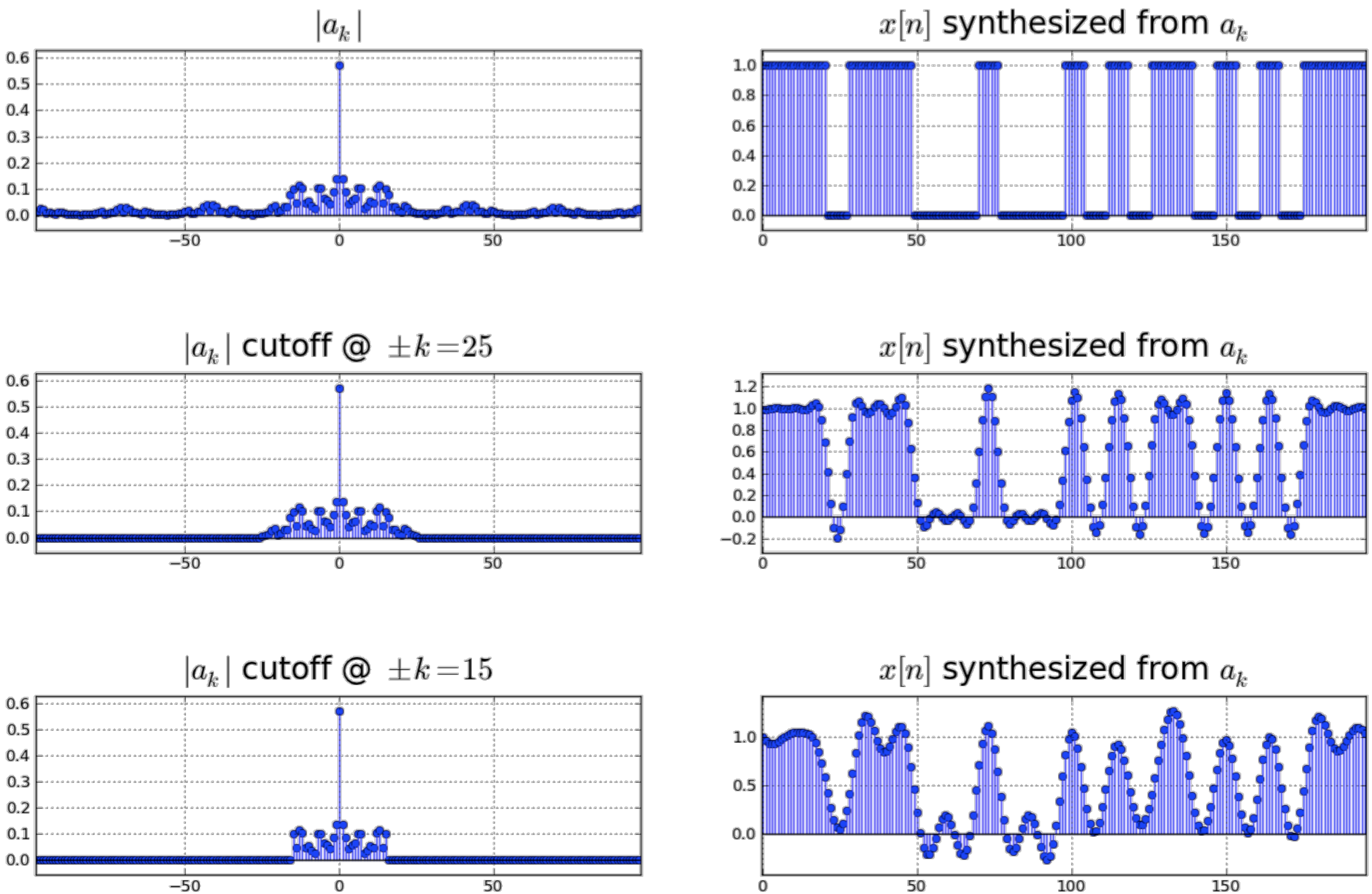
- The waveform x[n] cannot vary faster than the step change every 7 samples, so we expect the highest frequency components in the waveform to have a period around 14 samples. (The is rough and qualitative, as x[n] is not sinusoidal.)

- A period of 14 corresponds to a frequency of $2\pi/14 = \pi/7$, which is 1/7 of the way from 0 to the positive end of the frequency axis at π (so k approximately 100/7 or 14 in the figure). And that indeed is the neighborhood of where the Fourier coefficients drop off significantly in magnitude.

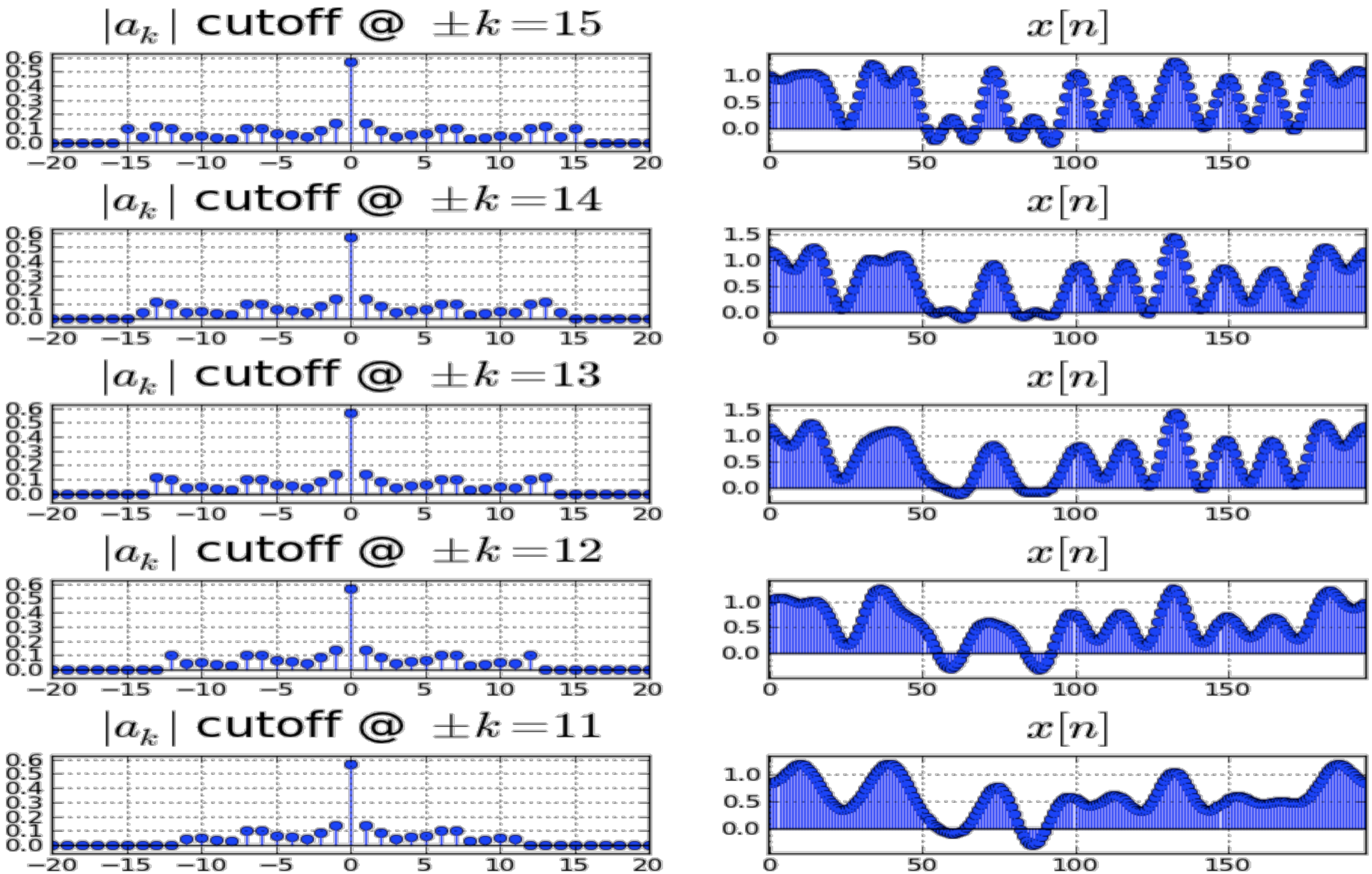
- There are also lower-frequency components corresponding to the fact that the 1 or 0 level may be held for several bit slots.

- And there are higher-frequency components that result from the transitions between voltage levels being sudden, not gradual.

Effect of Low-Pass Channel

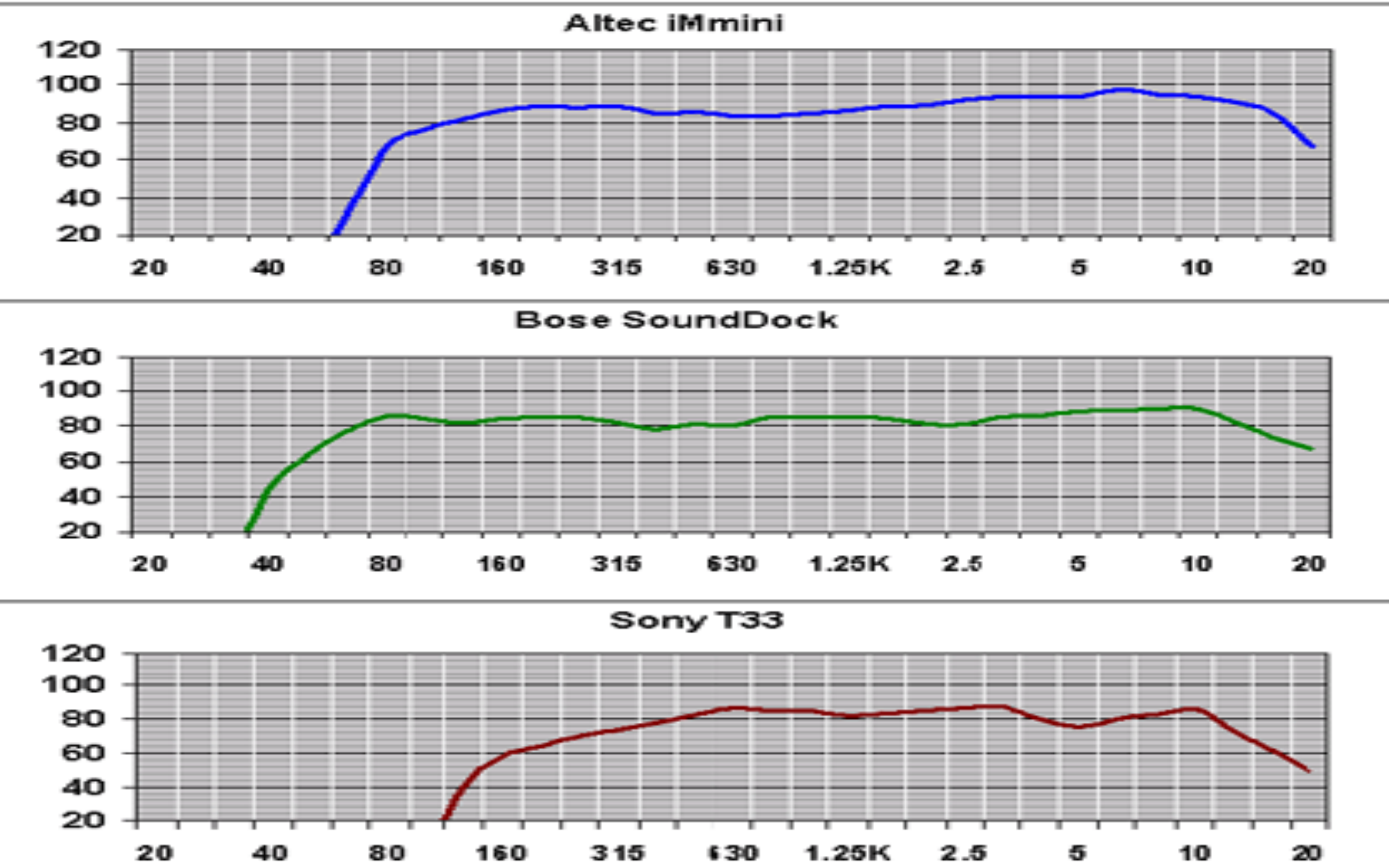


How Low Can We Go?



7 samples/bit → 14 samples/period → k=(N/14)=(196/14)=14

Back to our Audiocom lab example

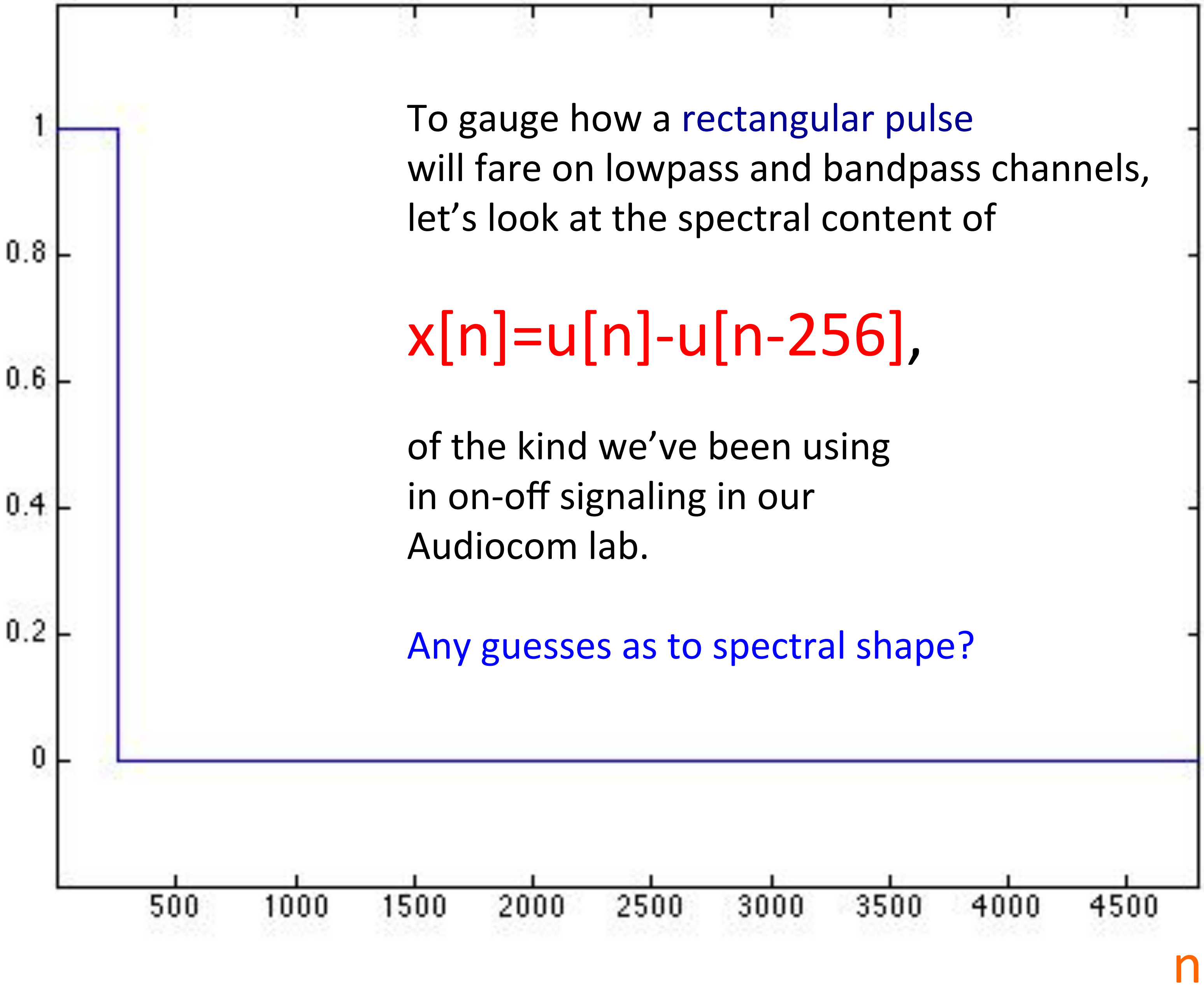


http://www.pcmag.com/article2/0,2817,1769243,00.asp

Phase of the frequency response is important too!

- Maybe not if we are only interested in audio, because the ear is not so sensitive to phase distortions

- But it's certainly important if we are using an audio channel to transmit non-audio signals such as digital signals representing 1's and 0's, not intended for the ear



To gauge how a rectangular pulse will fare on lowpass and bandpass channels, let's look at the spectral content of

$$x[n]=u[n]-u[n-256],$$

of the kind we've been using in on-off signaling in our Audiocom lab.

Any guesses as to spectral shape?

Derivation of DTFT for rectangular pulse x[m]=u[m]-u[m-N]

$$X(\Omega) = \sum_{m=0}^{N-1} x[m] e^{-j\Omega m}$$

$$= 1 + e^{-j\Omega} + e^{-j2\Omega} + \ldots + e^{-j\Omega(N-1)}$$

=> Height N at the origin.

$$= (1 - e^{-j\Omega N}) / (1 - e^{-j\Omega})$$

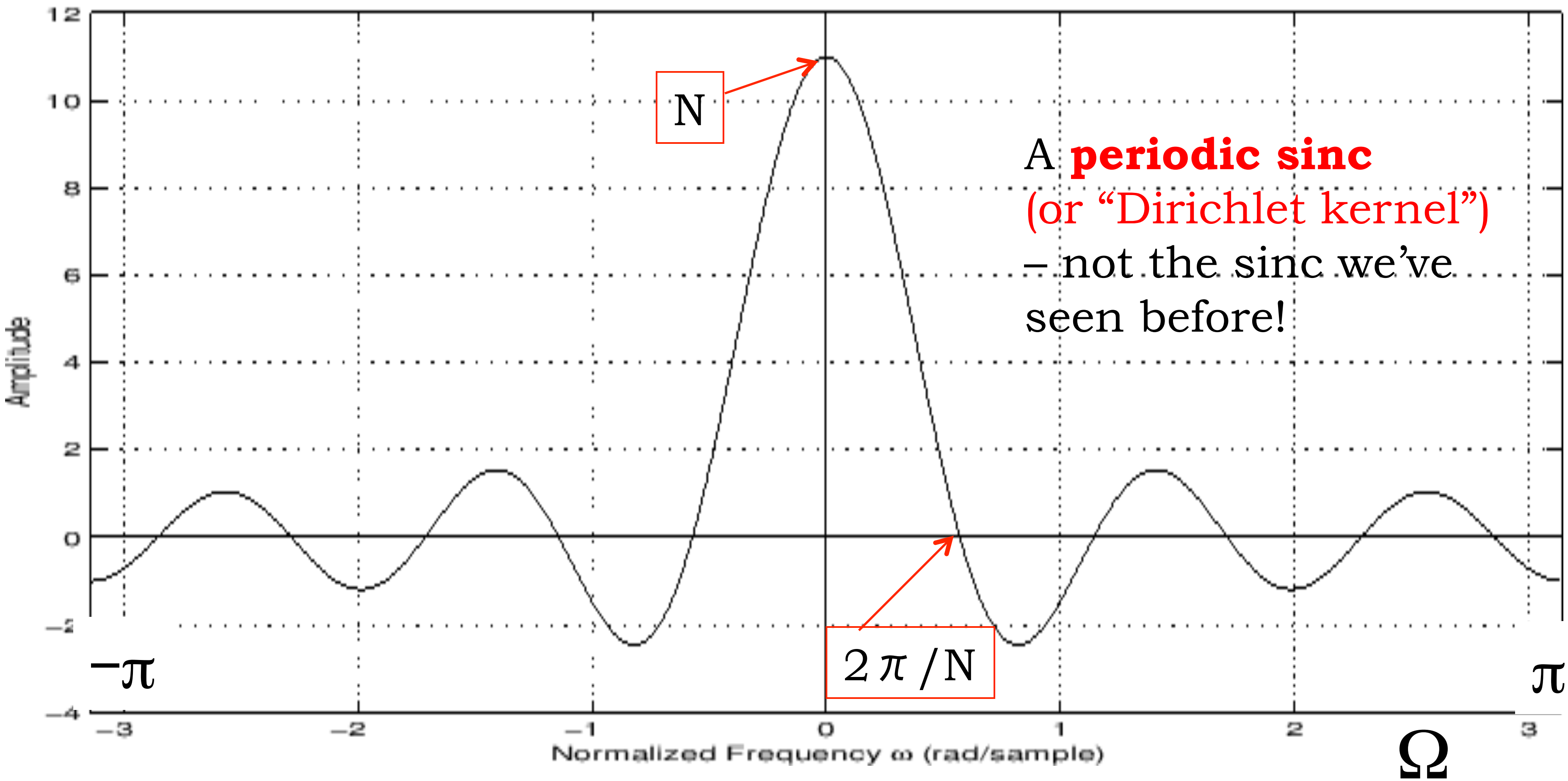
$$= e^{-j\Omega(N-1)/2} \frac{\sin(\Omega N/2)}{\sin(\Omega/2)}$$

First zero-crossing at $2\pi/N$

Shifting in time only changes the phase term in front.
If the rectangular pulse is centered at 0, this term is 1.

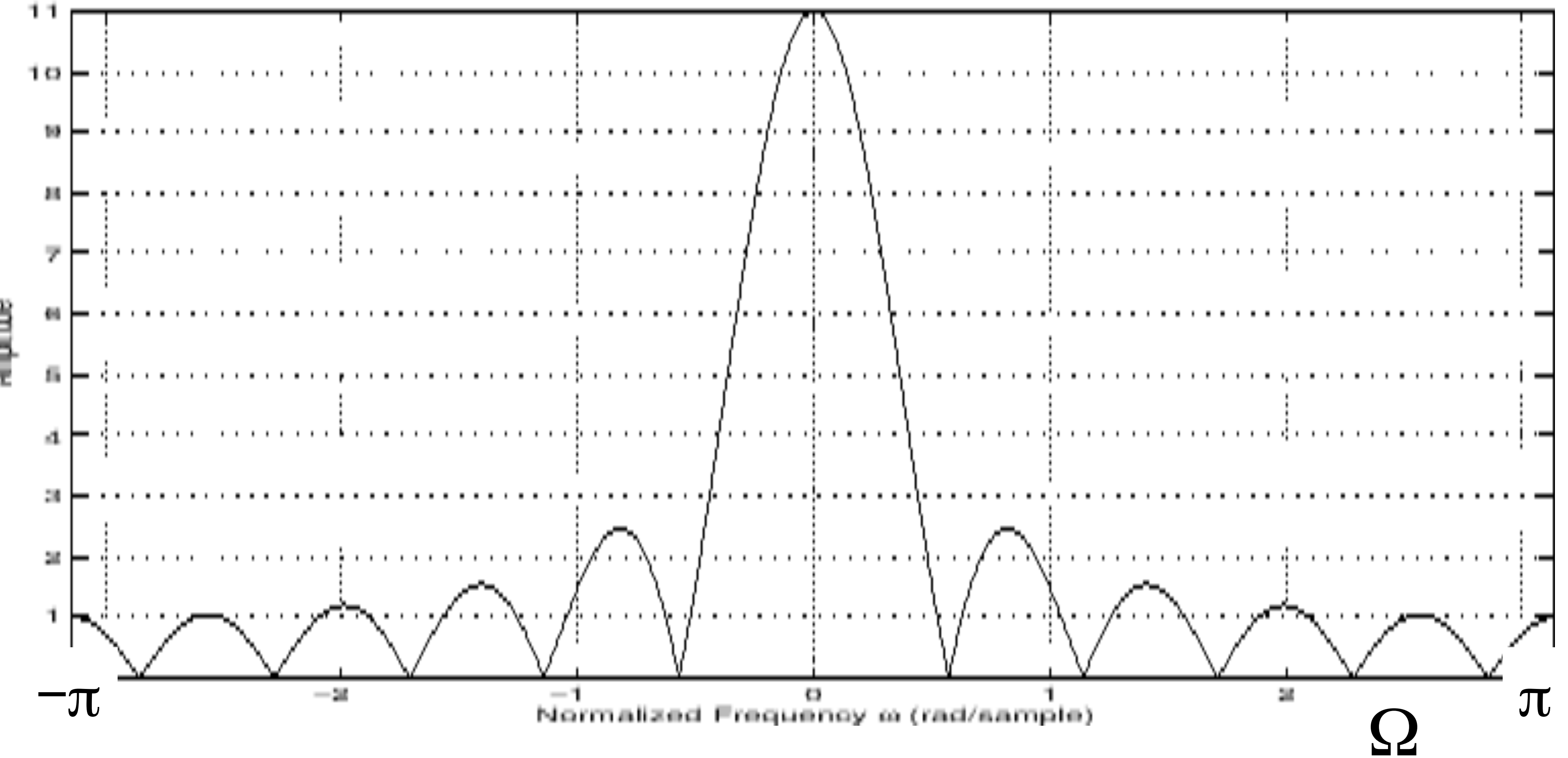
DTFT of x[n] = u[n+5] – u[n-6]

(centered rectangular pulse of length 11)

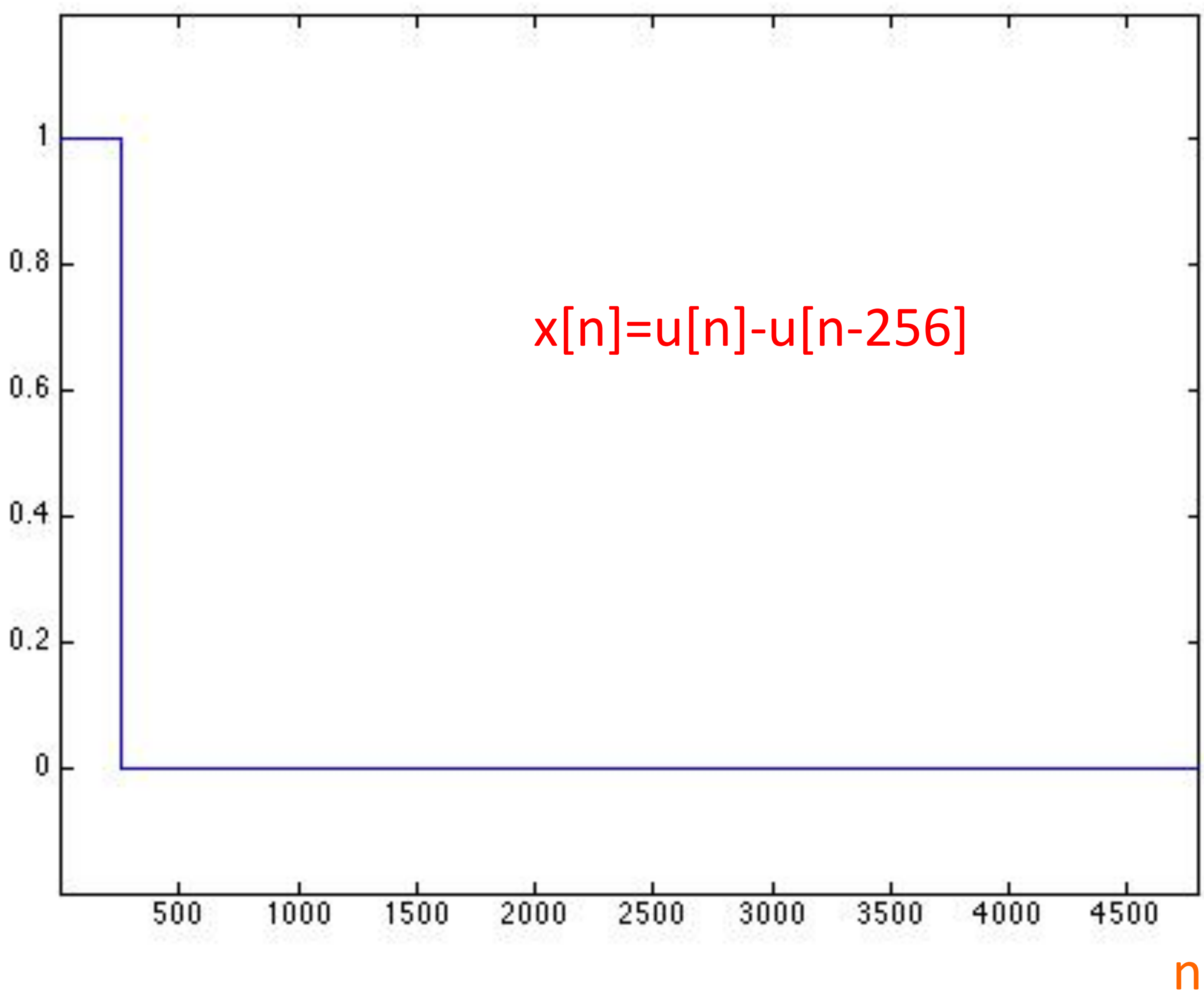


https://ccrma.stanford.edu/~jos/sasp/Rectangular_Window.html

Magnitude of preceding DTFT

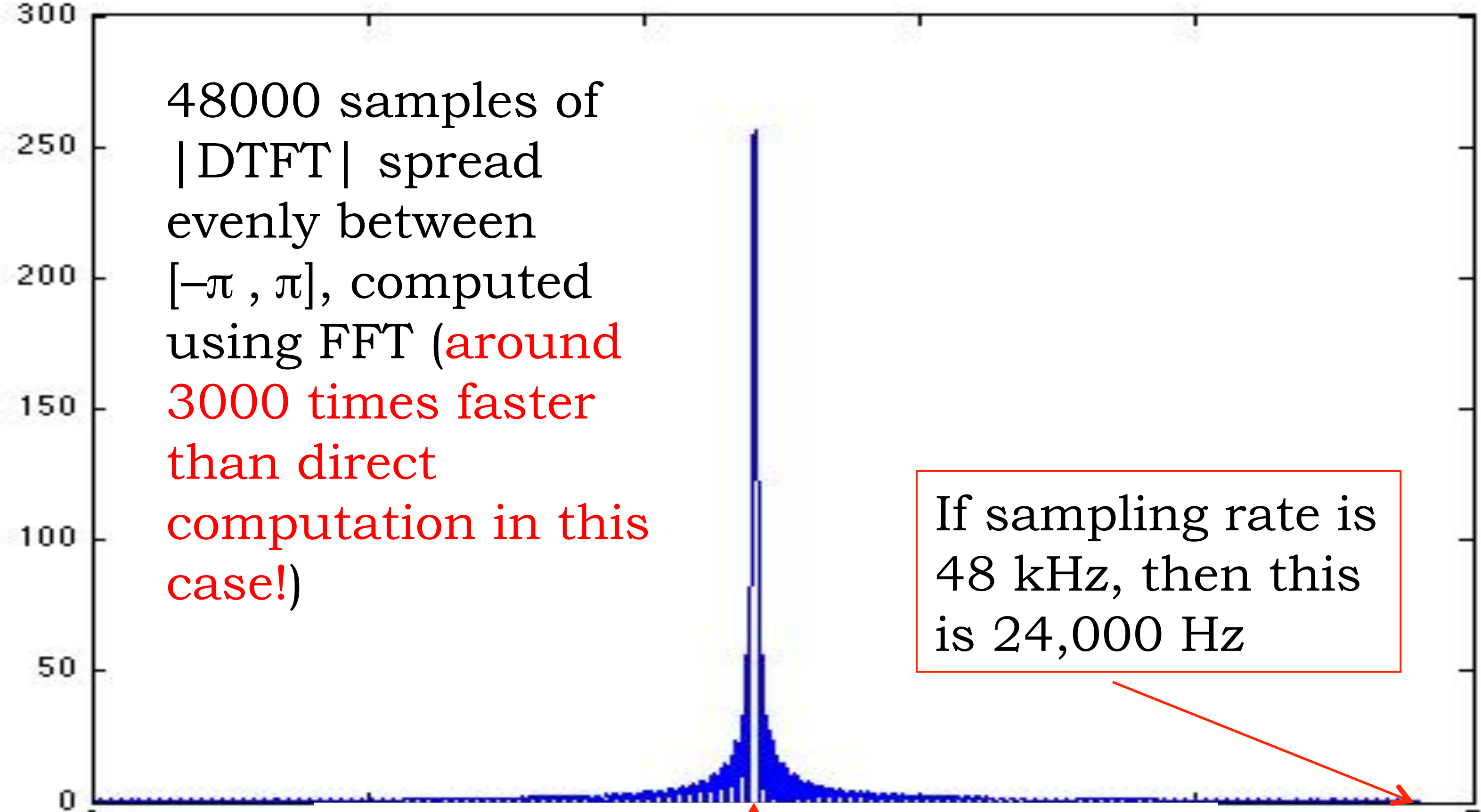


https://ccrma.stanford.edu/~jos/sasp/Rectangular_Window.html



1
0.8
0.6
0.4
0.2
0
500
1000
1500
2000
2500
3000
3500
4000
4500
x[n]=u[n]-u[n-256]
n

|DTFT| of x[n]=u[n]-u[n-256], rectangular pulse of length 256:



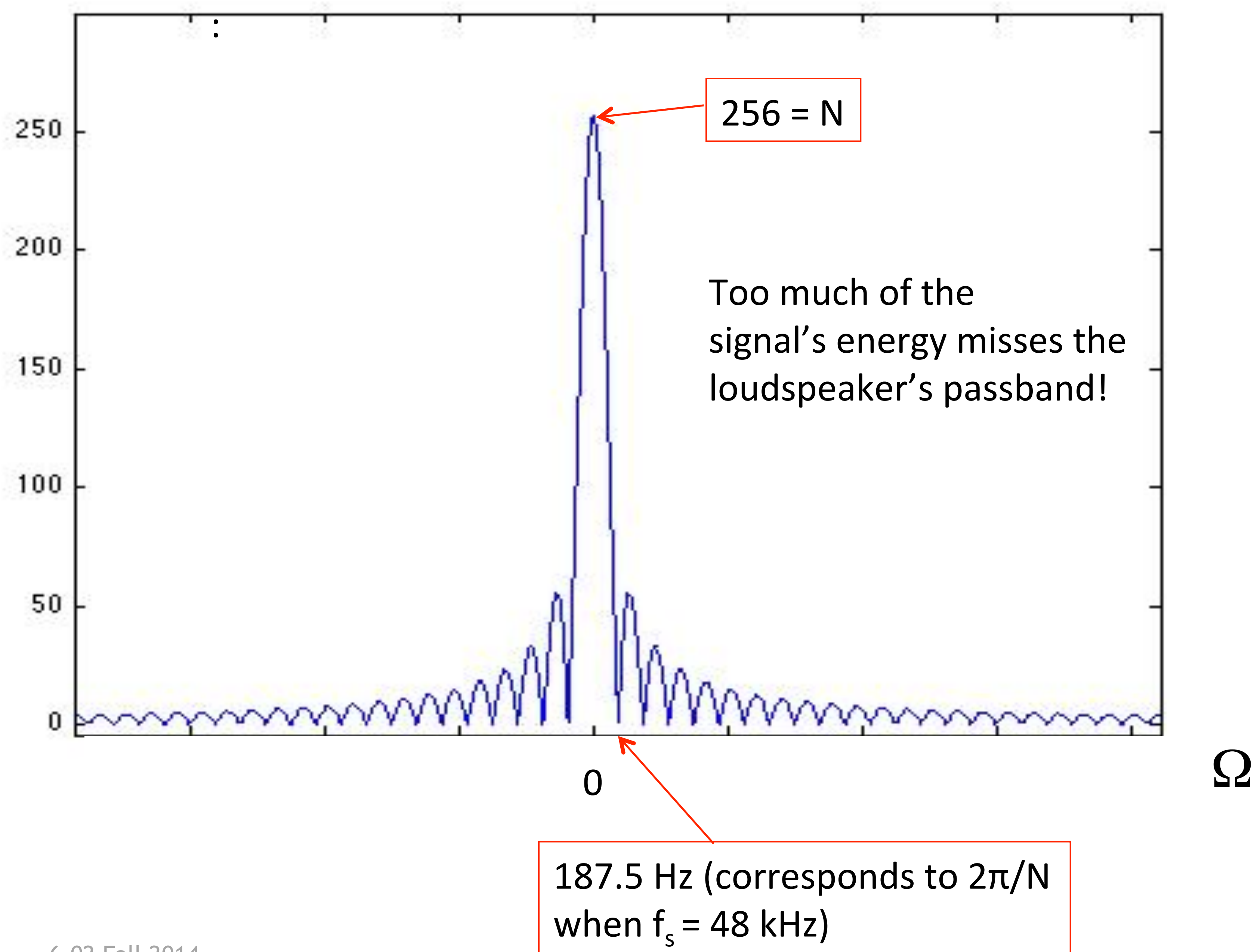
$-\pi$

$\Omega = \pi$

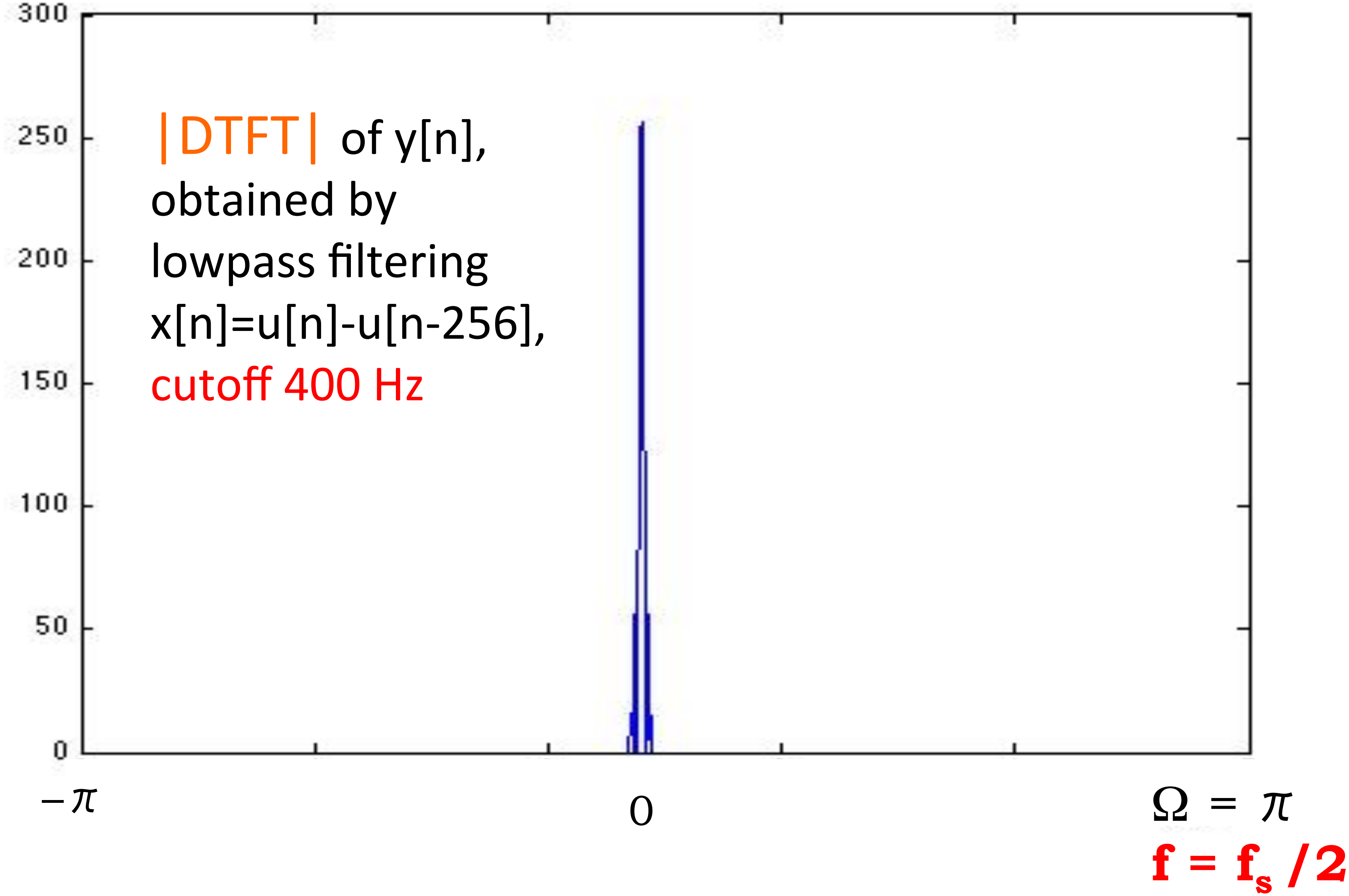
$f = f_s / 2$

0 rads/sample
0 Hz

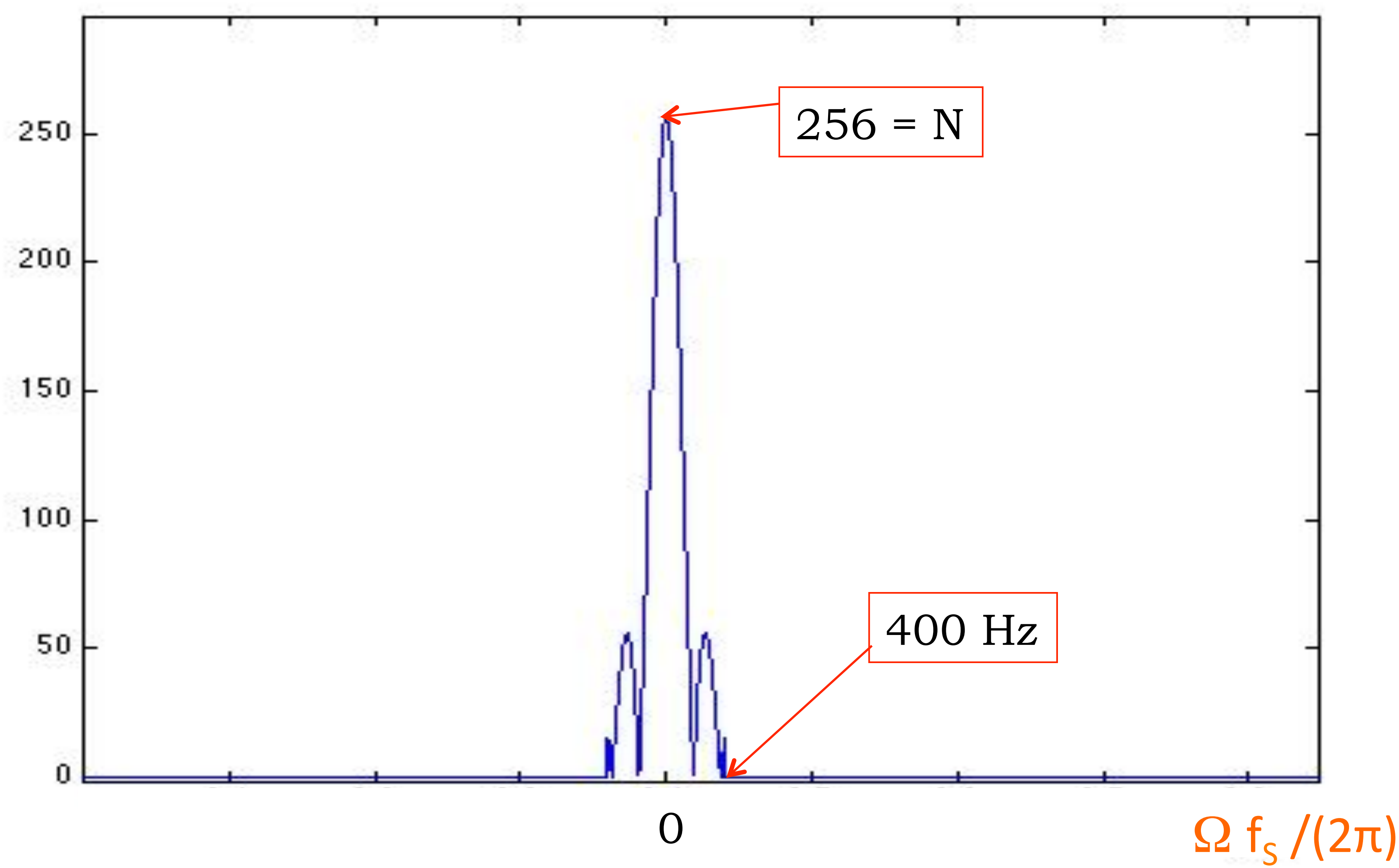
Zooming in on |DTFT| of x[n]=u[n]-u[n-256]



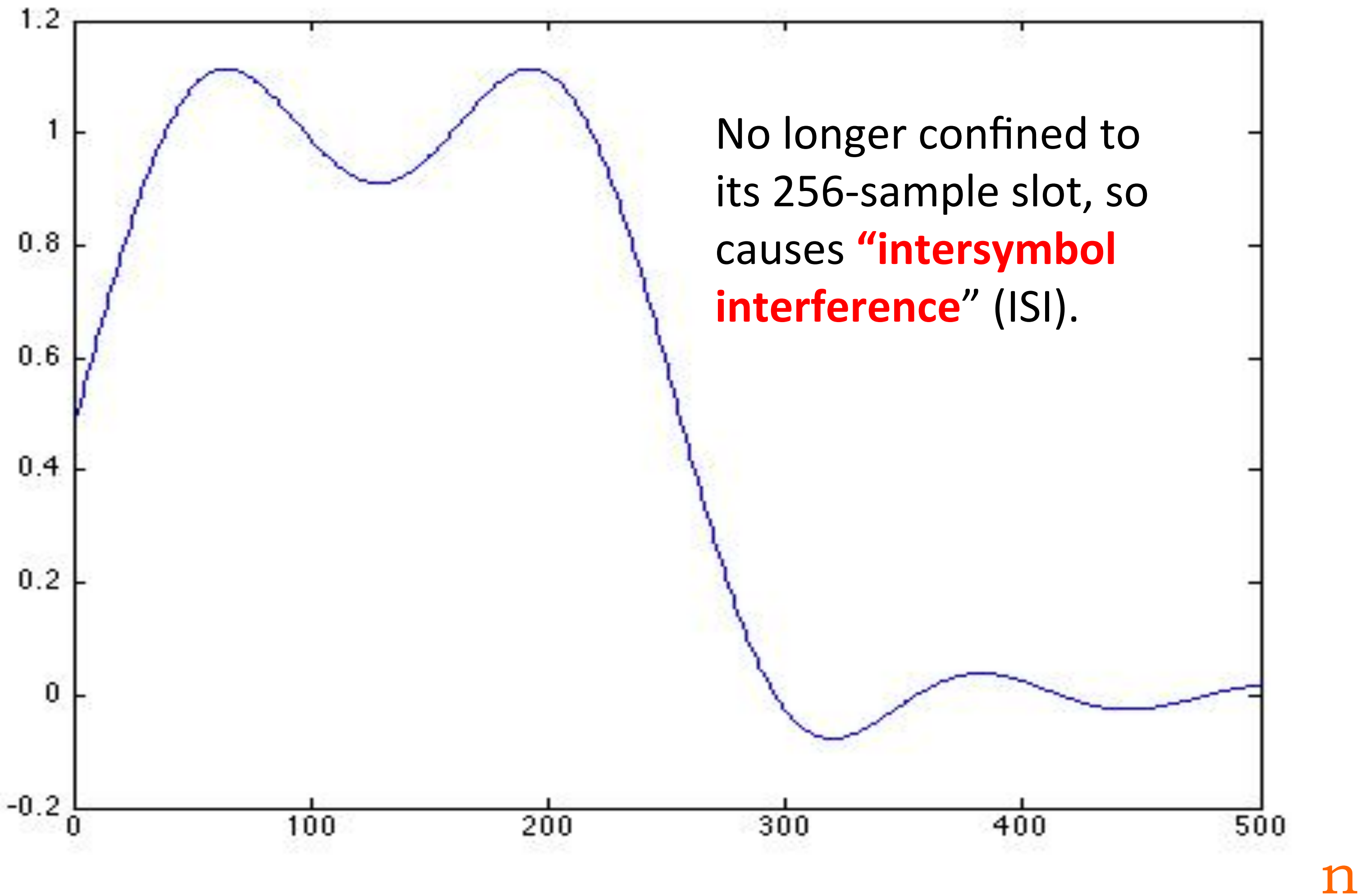
What if we sent this pulse through an ideal lowpass channel?



Zooming in on |DTFT| of y[n], obtained by lowpass filtering of x[n]=u[n]-u[n-256], cutoff 400 Hz



Corresponding output pulse y[n], obtained by lowpass filtering the rectangular pulse x[n]

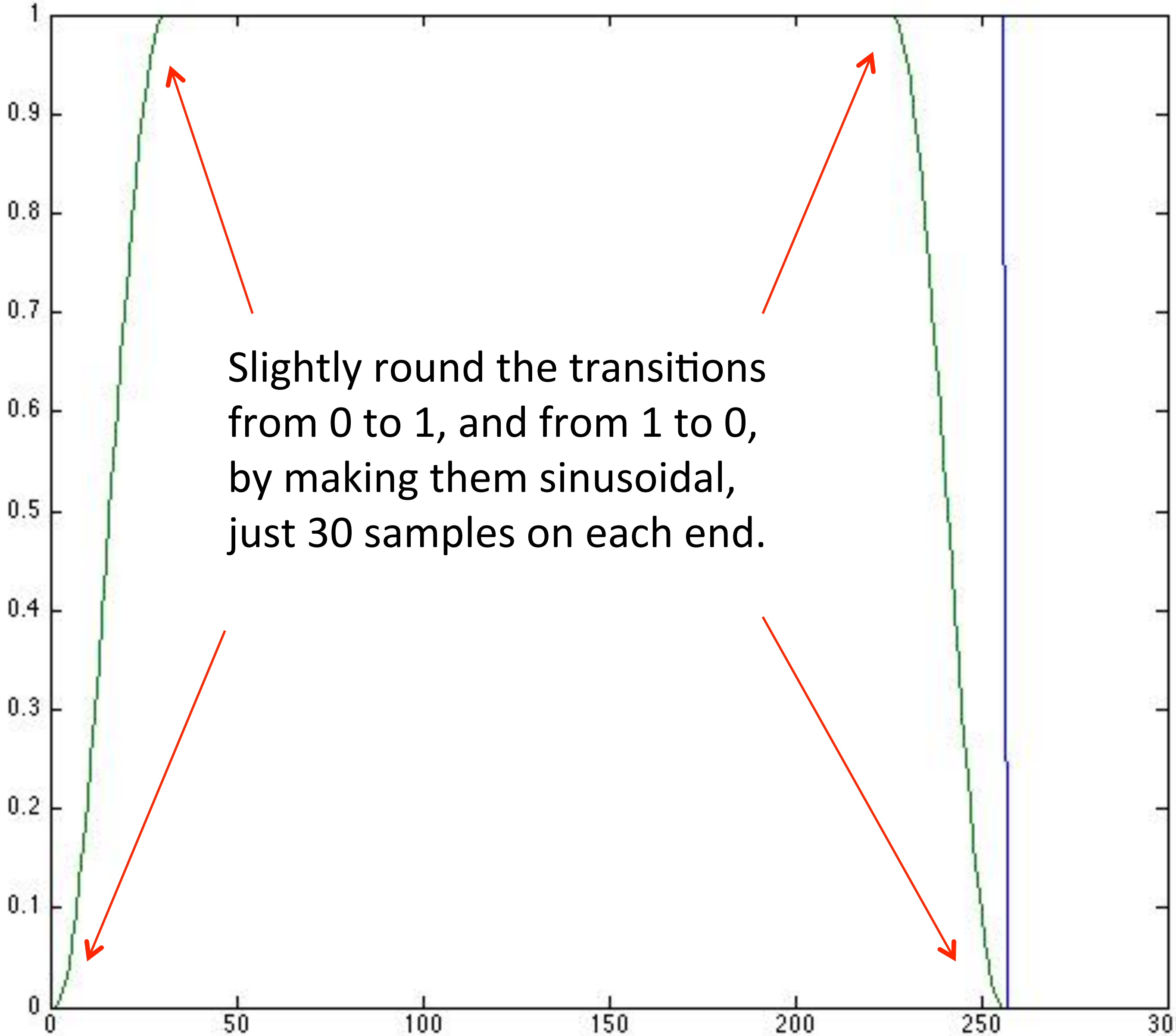


No longer confined to its 256-sample slot, so causes **"intersymbol interference"** (ISI).

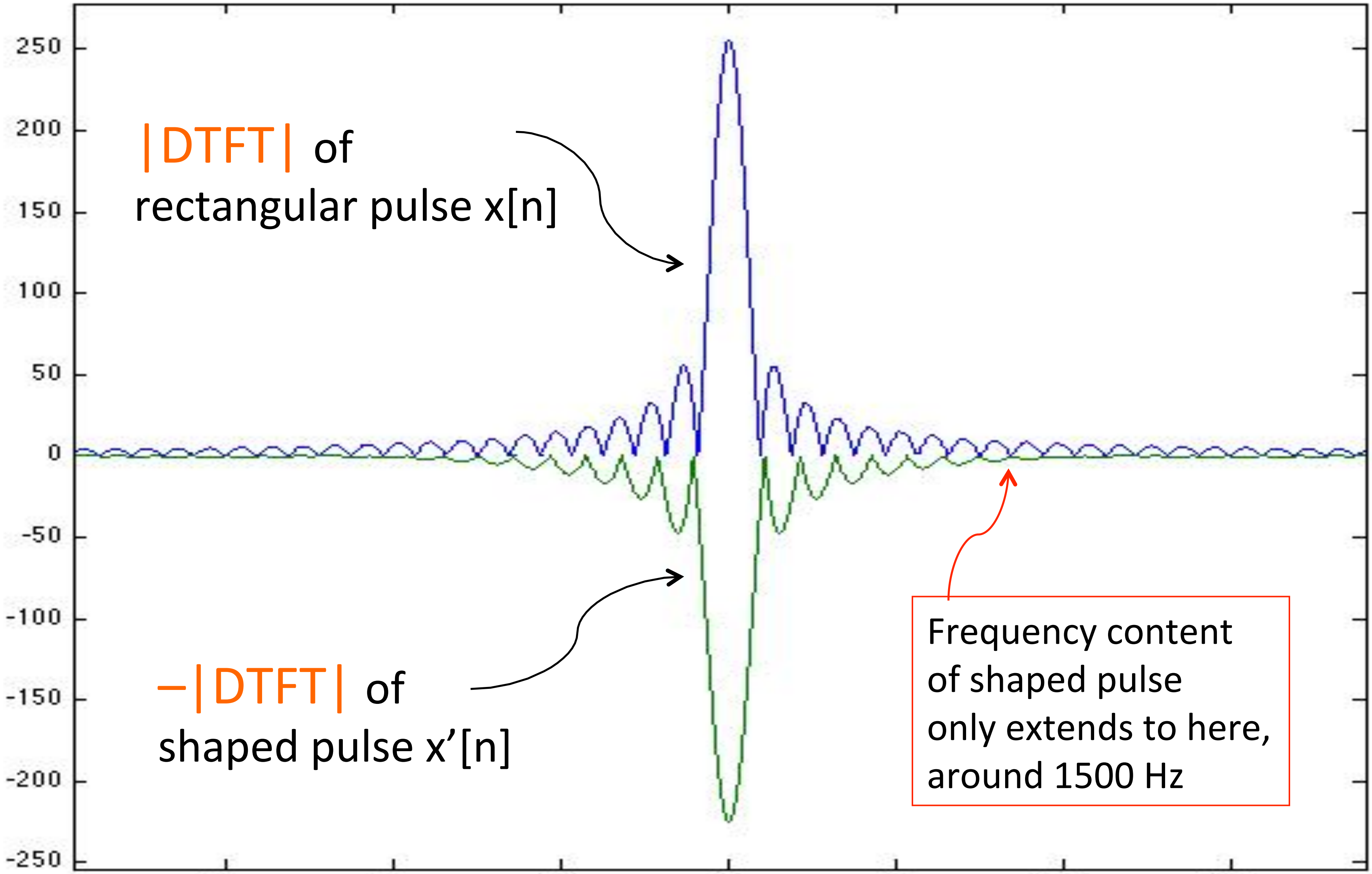
Complementary/dual behavior in time and frequency domains

- Wider in time, narrower in frequency; and vice versa.
  - This is actually the basis of the uncertainty principle in physics!
- Smoother in time, sharper in frequency; and vice versa
- Rectangular pulse in time is a (periodic) sinc in frequency, while rectangular pulse in frequency is a sinc in time; etc.

A shaped pulse x’[n] versus a rectangular pulse x[n]

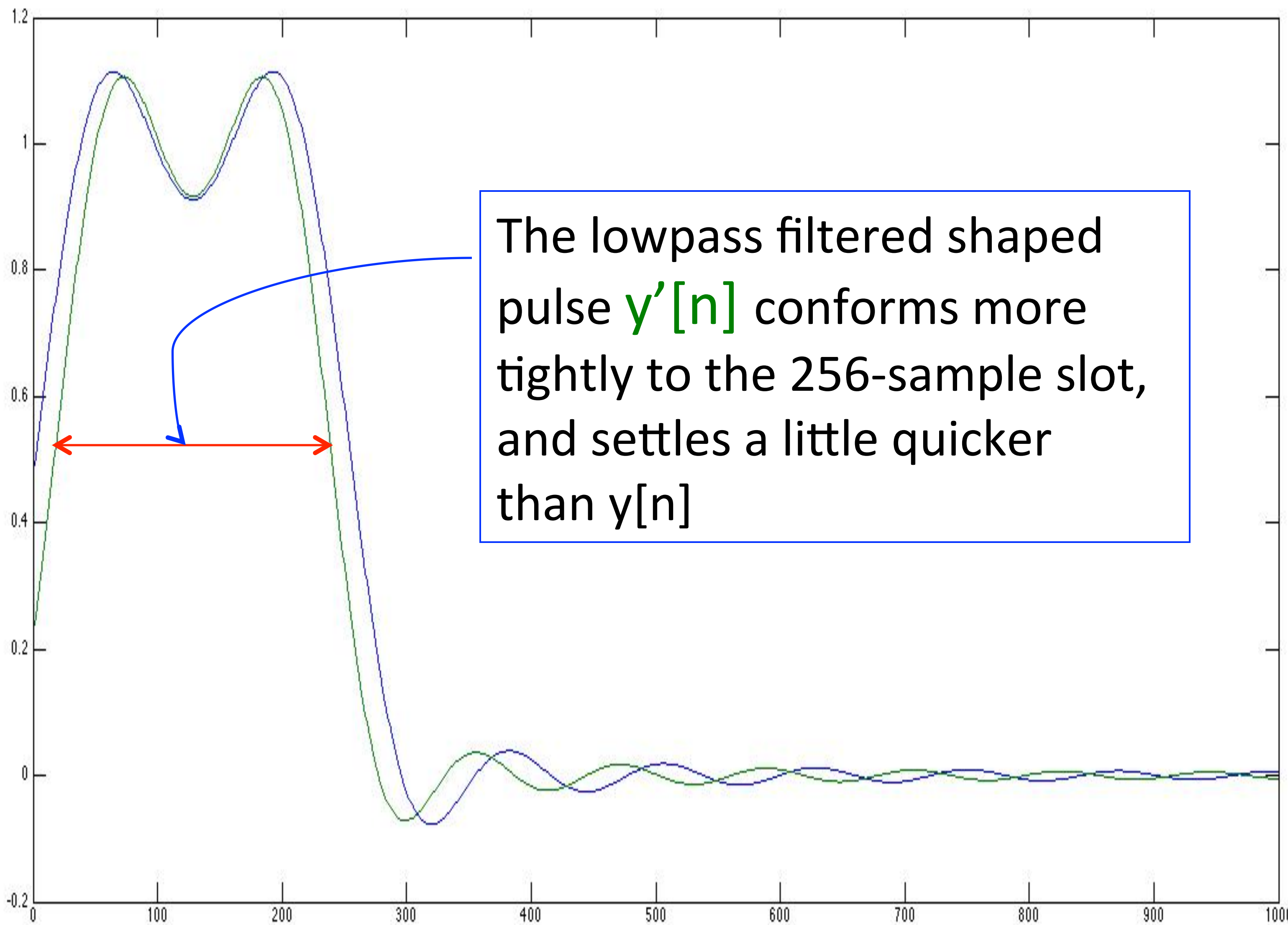


In the spectral domain:



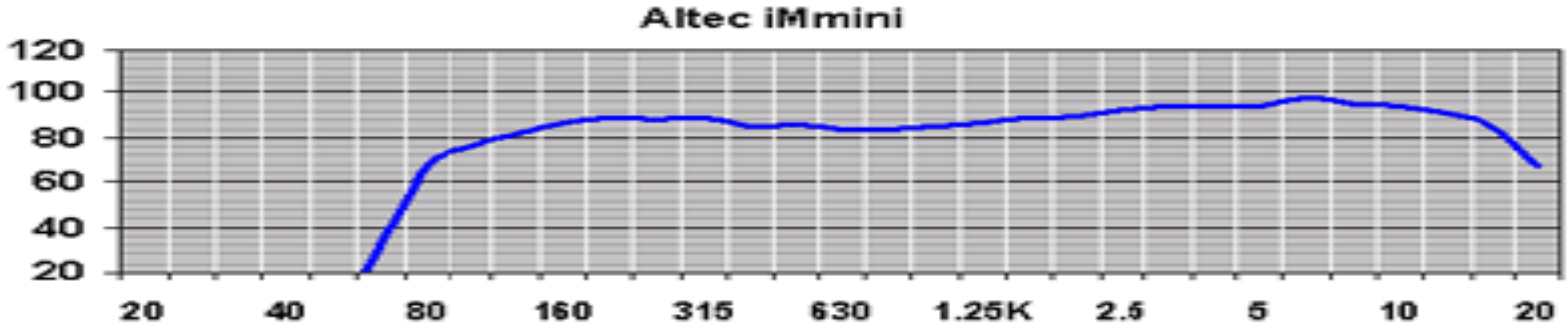
$\Omega f_s / (2\pi)$

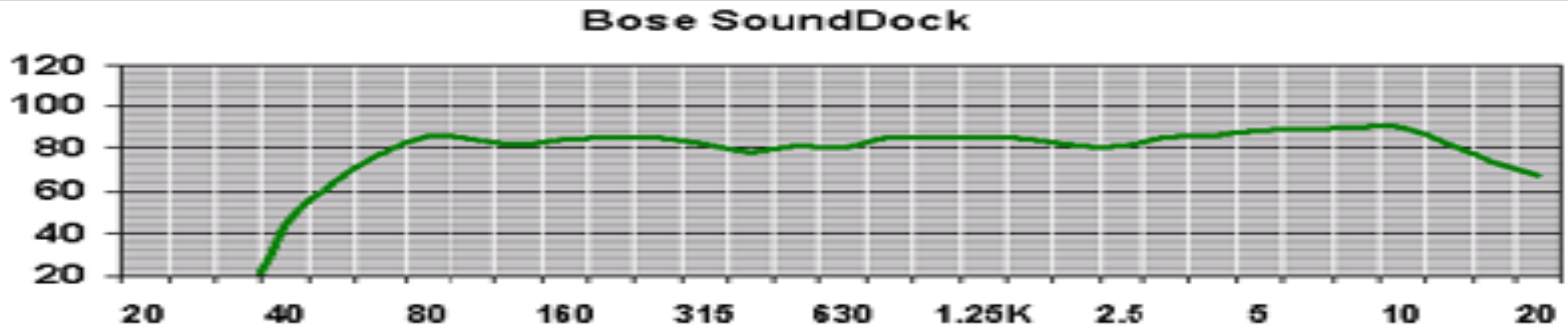
After passing the two pulses through a 400 Hz cutoff lowpass filter:

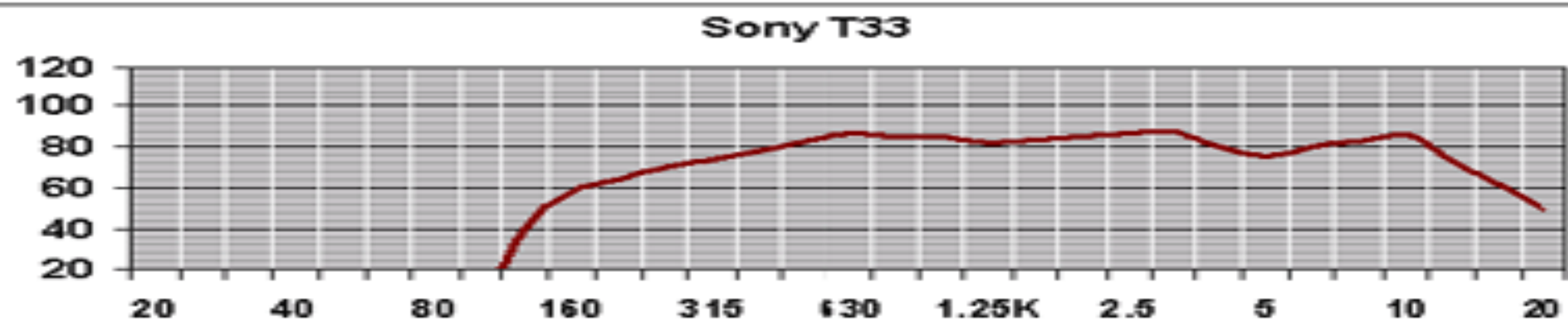


The lowpass filtered shaped pulse y'[n] conforms more tightly to the 256-sample slot, and settles a little quicker than y[n]

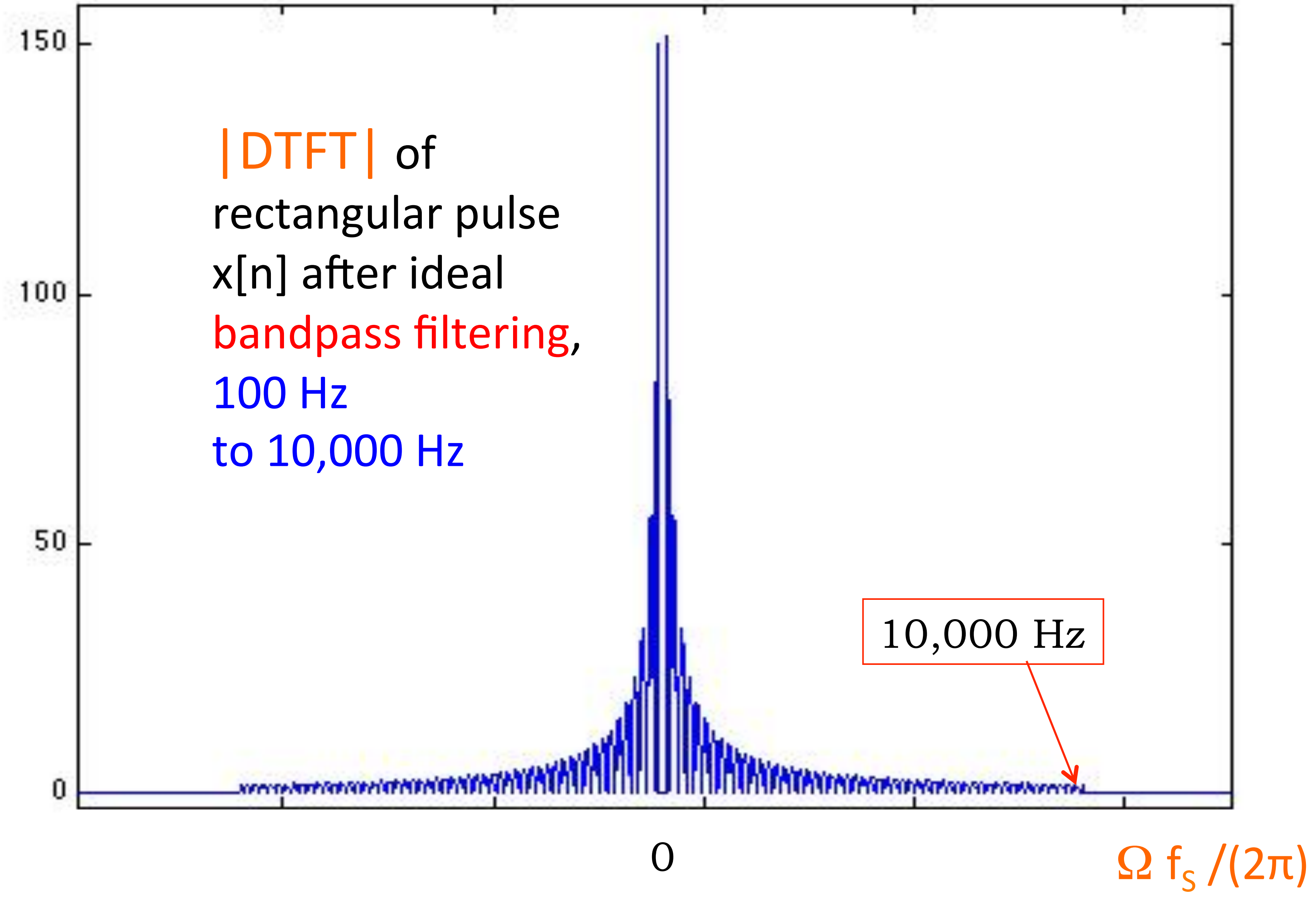
But loudspeakers are bandpass, not lowpass



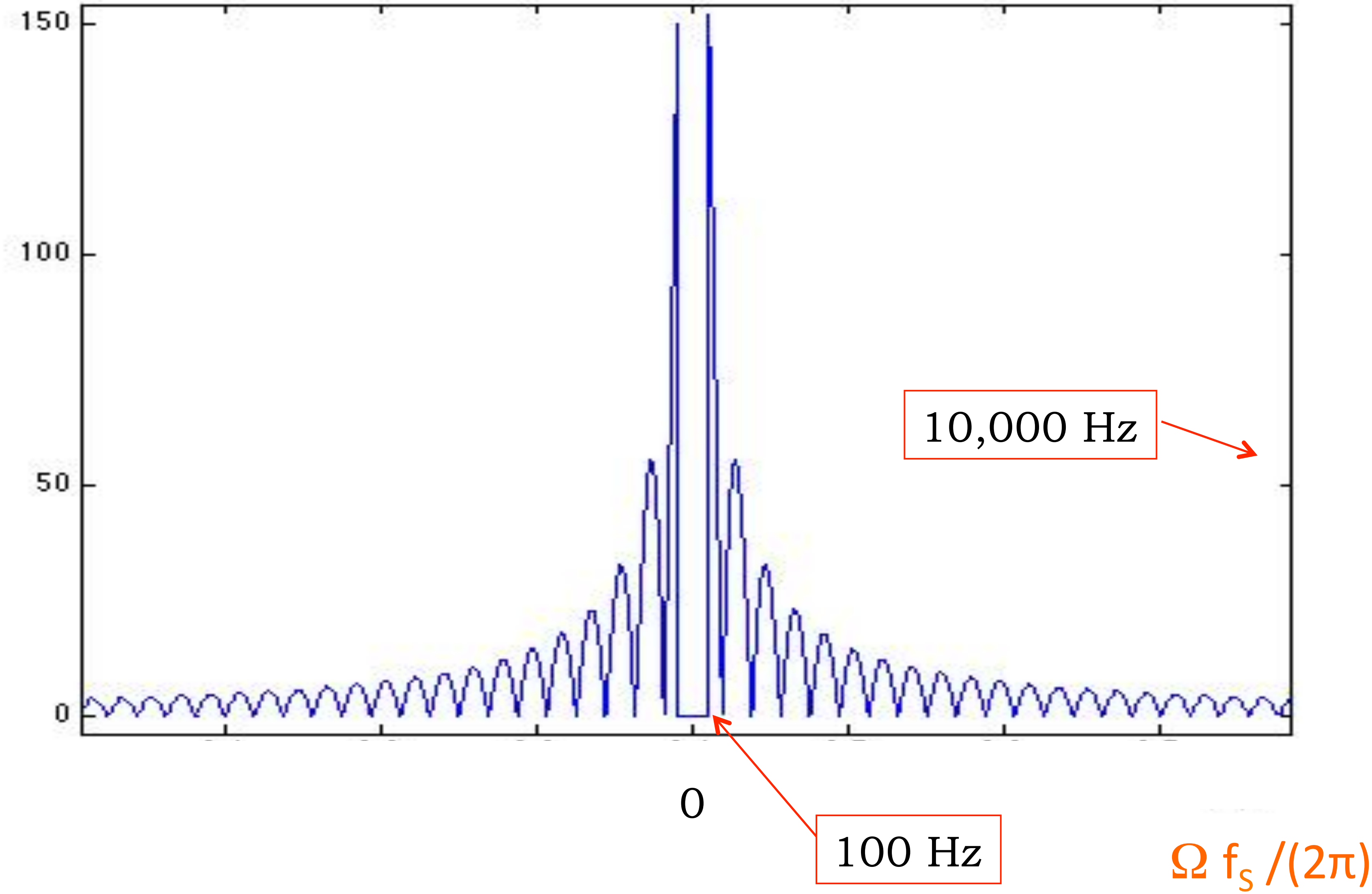


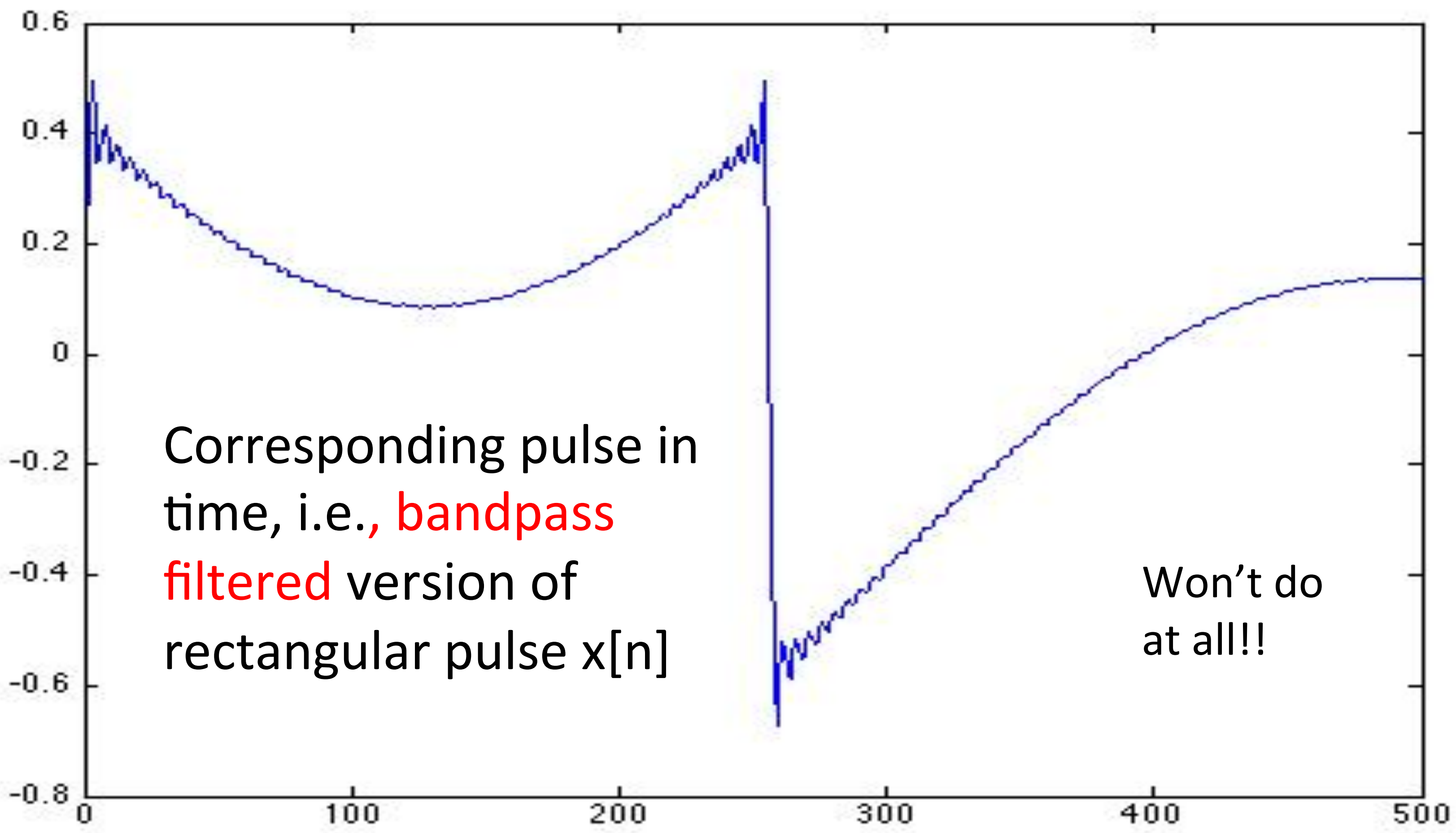


http://www.pcmag.com/article2/0,2817,1769243,00.asp


150
100
50
0
|DTFT| of rectangular pulse x[n] after ideal bandpass filtering, 100 Hz to 10,000 Hz
10,000 Hz
0
Ω f_S /(2π)

Zooming in:




0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
0
100
200
300
400
500
Corresponding pulse in time, i.e., bandpass filtered version of rectangular pulse x[n]
Won't do at all!!

The Solution: Modulation

- Shift the spectrum of the signal x[n] into the loudspeaker's passband by **modulation!**

- The basic idea is Fessenden's heterodyne principle:

(sinusoid at frequency f_1) **x** (sinusoid at frequency f_2)
= (sinusoids at frequency f_1-f_2) **+** (sinusoid at frequency f_1+f_2)

i.e., multiplying sinusoids yields the
sum and difference frequencies, because

$$e^{j(\pm\Omega_1)n}e^{j(\pm\Omega_2)n} = e^{j(\pm\Omega_1\pm\Omega_2)n}$$

The Solution: Modulation

- Shift the spectrum of the signal x[n] into the loudspeaker's passband by **modulation!**

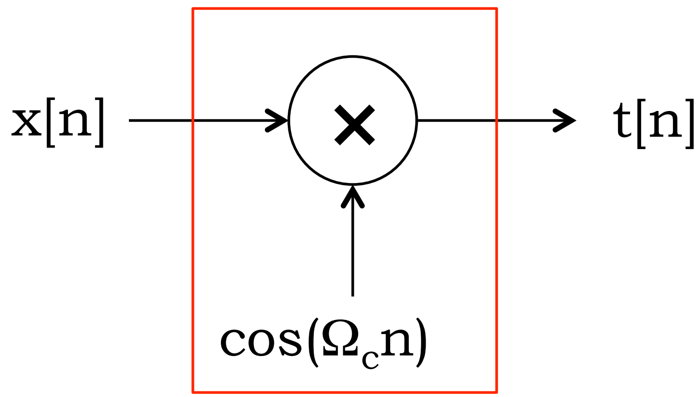
$$x[n]\cos(\Omega_c n) = 0.5x[n](e^{j\Omega_c n} + e^{-j\Omega_c n})$$

$$= \frac{0.5}{2\pi}[\int_{<2\pi>} X(\Omega')e^{j(\Omega'+\Omega_c)n}d\Omega' + \int_{<2\pi>} X(\Omega'')e^{j(\Omega''-\Omega_c)n}d\Omega'']$$

$$= \frac{0.5}{2\pi}[\int_{<2\pi>} X(\Omega-\Omega_c)e^{j\Omega n}d\Omega + \int_{<2\pi>} X(\Omega+\Omega_c)e^{j\Omega n}d\Omega]$$

Spectrum of modulated signal comprises half-height replications of X(Ω) centered as $\pm\Omega_c$ (i.e., plus and minus the carrier frequency). So choose carrier frequency comfortably in the passband, leaving room around it for the spectrum of x[n].

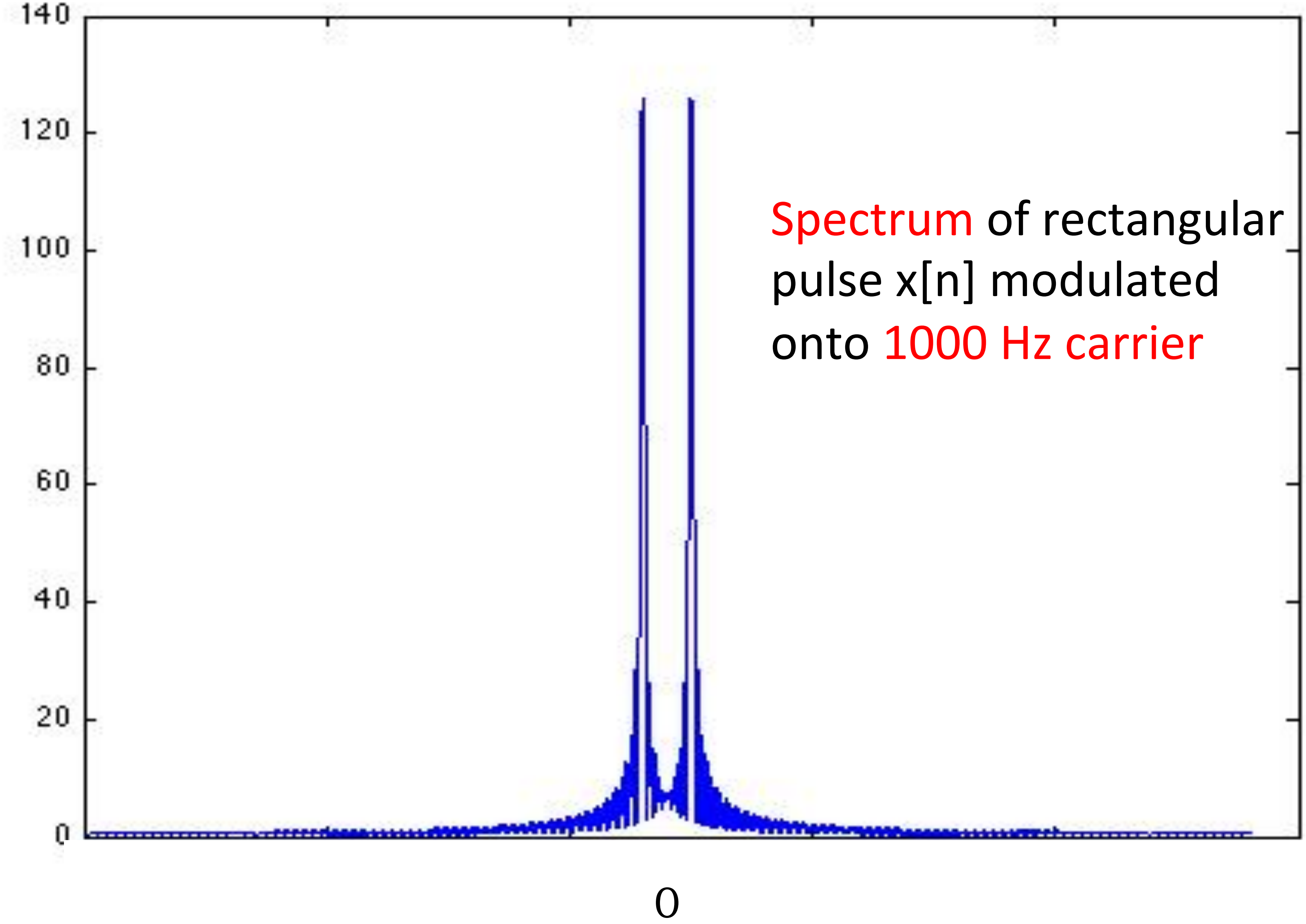
Is Modulation Linear? Time-Invariant?



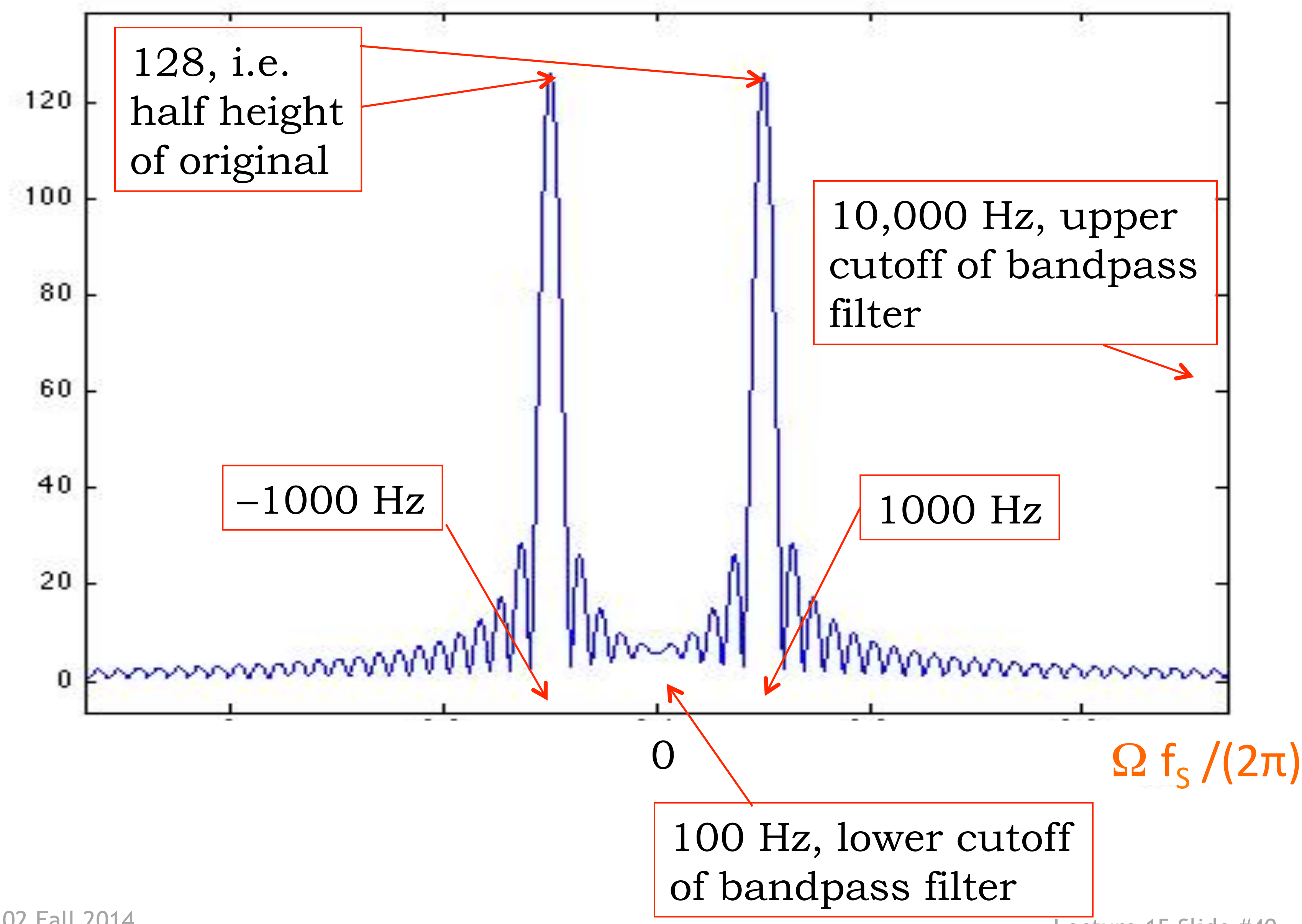
… as a system that takes input x[n] and produces output t[n] for transmission?

Yes, linear!

No, not time-invariant!

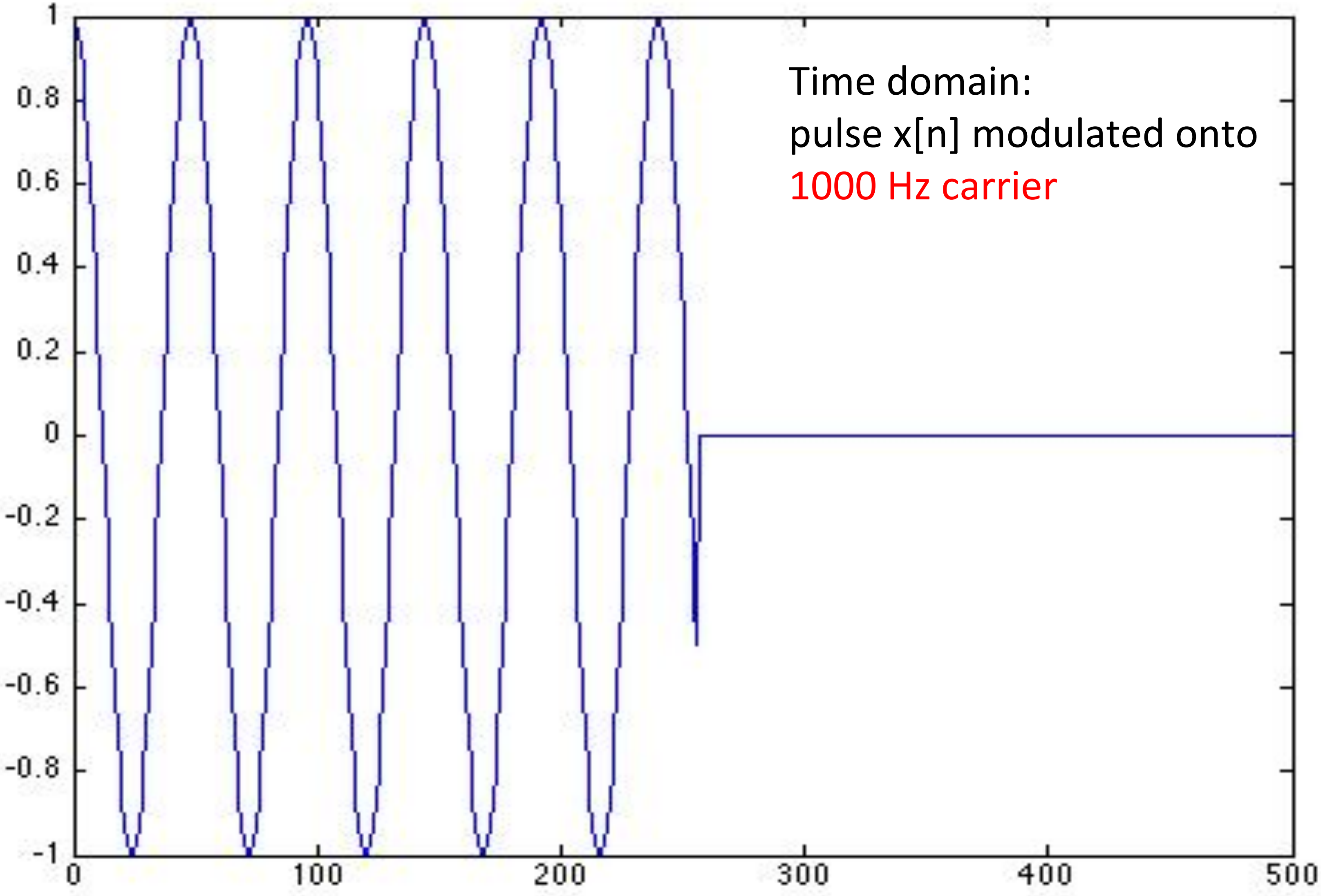

140
120
100
80
60
40
20
0
Spectrum of rectangular pulse x[n] modulated onto 1000 Hz carrier
0

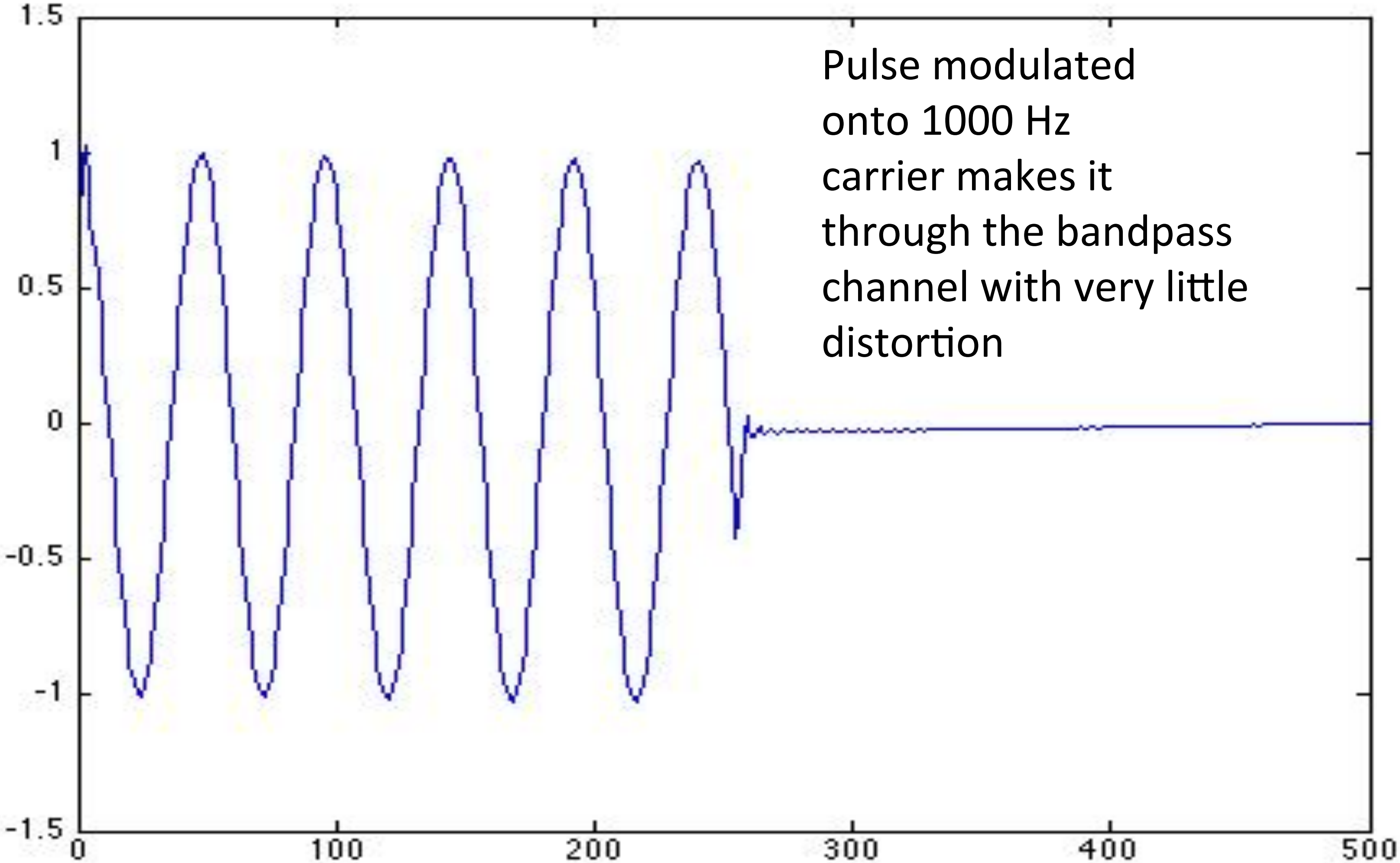
Zooming in:



120
100
80
60
40
20
0
128, i.e. half height of original
10,000 Hz, upper cutoff of bandpass filter
−1000 Hz
1000 Hz
0
Ω f_S /(2π)
100 Hz, lower cutoff of bandpass filter

So for our rectangular pulse example:




1.5
1
0.5
0
-0.5
-1
-1.5
0
100
200
300
400
500
Pulse modulated onto 1000 Hz carrier makes it through the bandpass channel with very little distortion

At the Receiver: Demodulation

- In principle, this is (as easy as) modulation again:

If the received signal is

$$r[n] = x[n]\cos(\Omega_c n),$$

then simply compute

$$\begin{aligned} d[n] &= r[n]\cos(\Omega_c n) \\ &= x[n]\cos^2(\Omega_c n) \\ &= 0.5\,\{x[n] + x[n]\cos(2\Omega_c n)\} \end{aligned}$$

- What does the spectrum of d[n] look like?
- What constraint on the bandwidth of x[n] is needed for perfect recovery of x[n] by lowpass filtering of d[n]?